Self-similar sets with arbitrary Hausdorff and box-counting dimension

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Abstract

Hausdorff and box-counting are two of the most common notions of dimension. We describe a way to calculate a different dimension, called dilation dimension, for certain figures exhibiting self-similarity using simple transformations of the plane. We show that dilation dimension is equivalent to Hausdorff dimension and box-counting dimension for subsets of $\mathbb{R}^n$ for which it is defined, and use this equivalence to give another proof that Hausdorff dimension and box-counting dimension can take on any positive real value by constructing a simple generalized Cantor set. These concepts have all been described before, and the purpose of this note is to describe a way to show that dimension can take on any positive real value.

1 Introduction

The ordinary notion of dimension states that points in Euclidean space have dimension 0, lines have dimension 1, planes have dimension 2, and so on. A natural question is whether this idea can be extended so that the dimension of less well-behaved sets such as fractals may also be calculated. In many classes, fractals are introduced as examples of sets with non-integral dimension. It is thus natural to ask what possible values can be attained. The purpose of this short note is to describe one way to show any positive number is possible, for use in such classroom discussions. References for Section 1 can be found in [1].

Definition 1 (Box-counting dimension) *Given a set $F \subseteq \mathbb{R}^n$, the box-counting dimension of F is*

$$\dim_B F \;:=\; \lim_{\delta\to 0} \frac{\log N_\delta(F)}{-\log \delta} \tag{1}$$

where $N_\delta(F)$ is the smallest number of cubes of side length δ which cover F.

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Box-counting dimension is defined only for sets where the limit exists. The definition of Hausdorff dimension is more involved and requires us to first define several related terms. Let $|F|$ denote the diameter of a set $F \subseteq \mathbb{R}^n$, in other words

$$|F| \ := \ \sup\{d(x,y) : \ x, y \in F\} \tag{2}$$

where $d(x,y)$ is the Euclidean distance in $\mathbb{R}^n$.

Definition 2 (δ-cover) *Let $F \subseteq \mathbb{R}^n$. A countable (possibly finite) collection of sets $\{U_i\}$ is a δ-cover of F if $\{U_i\}$ covers F and each U_i has diameter at most δ.*

Defining δ-covers allows us to define the Hausdorff measure, an extension of the ordinary Lebesgue measure on $\mathbb{R}^n$.

Definition 3 (Hausdorff measure) *Let $F \subseteq \mathbb{R}^n$ and fix $s \geq 0$. For each $\delta > 0$, we define*

$$\mathcal{H}^s_\delta(F) \ = \ \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \ \{U_i\} \textit{ is a } \delta\textit{-cover of } F\right\} \tag{3}$$

The s-dimensional Hausdorff measure on F is defined as

$$\mathcal{H}^s(F) \ := \ \lim_{\delta \to 0} \mathcal{H}^s_\delta(F). \tag{4}$$

It can be shown that $\mathcal{H}^s$ is in fact a measure which is equivalent to the Lebesgue measure up to multiplication by a constant. Finally, we can define Hausdorff dimension.

Definition 4 (Hausdorff dimension) *Let $F \subseteq \mathbb{R}^n$. The Hausdorff dimension of F is*

$$\dim_H F \ := \ \inf\{s \geq 0 : \ \mathcal{H}^s(F) < \infty\}. \tag{5}$$

Hausdorff dimension is defined for any $F \subseteq \mathbb{R}^n$, making it a very useful measure of dimension. However, it can be difficult to calculate as seen from its technical definition. The same applies for box-counting dimension. Another question is which values Hausdorff and box-counting dimensions can take, and developing a simple way to calculate these dimensions will make answering this question much easier.

2 Dilation Dimension

Definition 5 (Dilation) *Given a set $F \subseteq \mathbb{R}^n$, a dilation of F is a mapping $f_d : x \to d \cdot x$ for any real $d > 1$.*

Definition 6 (Translation) *Given a set $F \subseteq \mathbb{R}^n$, a translation of F is a mapping $T_\lambda : x \to x + \lambda$.*

Definition 7 (Dilation Dimension) *Let $F \subseteq \mathbb{R}^n$. Suppose there exists a dilation f_d of F by a factor of d such that there exists a set of translations $T_1, \ldots, T_m$ such that*

$$f_d(F) = \bigsqcup_{i=1}^{m} T_i(F), \tag{6}$$

where $\bigsqcup$ denotes a disjoint union. The dilation dimension of F is $\dim_D F := \log(m)/\log(d)$.

Note that this means that $d^{\dim_D F} = m$.

Definition 8 (Contraction) *A contraction S is a function $\mathbb{R}^n \to \mathbb{R}^n$ such that there exists $0 < d < 1$ with $|S(x) - S(y)| \leq d|x - y|$. A contraction is simple if the equality holds, i.e., if there exists $0 < d < 1$ such that $|S(x) - S(y)| = d|x - y|$. If S is simple, then d is called the ratio of S.*

Definition 9 (Attractor) *An iterated function system (IFS) is a finite family of contractions $\{S_1, \ldots, S_m\}$ in $\mathbb{R}^n$. A set $S \subseteq \mathbb{R}^n$ is an attractor of the IFS if*

$$F = \bigsqcup_{i=1}^{m} S_i(F). \tag{7}$$

Lemma 1 *Let $F \subseteq \mathbb{R}^n$. Suppose there exists a dilation f_d of F by a factor of d such that there exist a set of translations $T_1, \ldots, T_m$ such that (6) holds. Then there exists an IFS $\{S_1, \ldots, S_m\}$ with constant ratio d for which F is an attractor.*

Proof: Set $S_i = f_d^{-1} \circ S_i$. Then $\{S_1, \ldots, S_m\}$ is the desired IFS. QED

Theorem 1 ([1], Theorem 9.3) *Let F be the attractor for an IFS $\{S_1, \ldots, S_m\}$ of simple contractions with constant ratio d. Then the Hausdorff dimension $\dim_H$ and box-counting dimension $\dim_B$ of F is given by $\log(m)/\log(d)$; or equivalently $d^{\dim_B F} = d^{\dim_H F} = m$.*

The proof is fairly technical, as calculating Hausdorff and box-counting dimension directly from their definitions is fairly difficult. Their exact definitions can also be found in [1].

Corollary 1 *Dilation dimension is equivalent to Hausdorff dimension and box-counting dimension are equivalent for any set F for which it is defined.*

Proof: The proof follows directly from Lemma 1 and Theorem 1. QED

Theorem 2 *Let $r \geq 0$ be a nonnegative real number. There exists $n \in \mathbb{N}$ and a set $F \subseteq \mathbb{R}^n$ such that $\dim_H F = \dim_B F = r$.*

Proof: Since we have shown dilation dimension and Hausdorff dimension are equivalent, we can merely show that the statement holds for dilation dimension. First suppose $r = 0$. Take the empty set $\emptyset$. Dilating $\emptyset$ by any factor d gives 1 copy of $\emptyset$, so the dilation dimension is $\log_r 1 = 0$.

Now suppose r is a positive integer. The r-dimensional unit hypercube in $\mathbb{R}^r$ has Hausdorff (and ordinary) dimension r.

Now suppose $r > 0$ is not an integer. Let n be the smallest integer greater than or equal to r (the ceiling of r) and set $b = 2^{n/r}$. Since $n > r > 0$, $b > 2$. Define the set F as follows:

$$F = \left\{ (x_1, \ldots, x_n) : \forall 1 \leq k \leq n, \exists I_k \subseteq \mathbb{N}, x_k = \sum_{i \in I_k} b^{-i} \right\}.$$

Each x_k in the above is analogous to a number expressed in base b using only the digits 0 and 1, and between 0 and 1, inclusive. However, b is not necessarily an integer. We do have that $b > 2$, however, so each x_k can be represented uniquely. We claim that F has dilation dimension r. Dilate F by a factor of b. We now have 2^n copies of F, as we can disjointly partition bF into 2^n copies, each given by

$$F + \delta \ : \ \delta = \{(\delta_1, \ldots, \delta_n) : \delta_i \in \{0, 1\}\}.$$

There are 2^n such δs. Now we have that the dilation dimension of F satisfies

$$b^{\dim_D F} = 2^n.$$

Solving for b gives $b = 2^{n/\dim_D F}$, so it is clear that $\dim_D F = r$ since we also have that $b = 2^{n/r}$ by definition. It then follows from Corollary 1 that $\dim_H F = \dim_B F = r$. QED

References

[1] Falconer, K., "Fractal Geometry: Mathematical Foundations and Applications", Wiley, 2013.

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