Lie Algebras and their Representations

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1 Motivation

We begin with some motivation. Let G be a finite group. A representation of G is a group homomorphism.

$$\rho : G \to \mathrm{GL}_n(\mathbb{C}) \tag{1.1}$$

We have that

1. Every representation is a direct sum of irreducible representations.
2. The number of irreducible representations equals the number of conjugacy classes.
3. The representation ρ is uniquely determined by its character $\chi_\rho = \mathrm{Tr} \circ \rho$.

We want to generalize this to infinite groups. The issue is most infinite groups are messy. But there are nice ones, in particular infinite groups with extra structure are often nice.

For example, putting a smooth manifold structure on a group gives you a Lie group. Some examples of Lie groups are $\mathrm{GL}_n(\mathbb{R}), \mathrm{SL}_n(\mathbb{R}), \mathrm{SO}_n(\mathbb{R})$, and so on. A representation of a Lie group is a *smooth* homomorphism $G \to \mathrm{GL}_n(\mathbb{C})$. Representations of Lie groups have applications in many areas such as physics, differential geometry, harmonic analysis (automorphic forms), number theory, algebraic geometry, etc.

Classifying these things seems hard, but the key insight is as follows. Let $\mathfrak{g} = T_e G$ be the tangent space of G at the origin. This is an $\mathbb{R}$-vector space, and the group structure on G induces a *Lie bracket* on $\mathfrak{g}$:

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \tag{1.2}$$

which satisfies the axioms of a Lie algebra.

Miracle! We have that $(\mathfrak{g}, [\cdot, \cdot])$ remembers almost everything about G! So in many cases instead of studying G we can study the Lie algebra $\mathfrak{g}$. Precisely, we have that there is a bijection between connected, simply connected Lie groups and Lie algebras over $\mathbb{R}$ given by taking the tangent space over the origin.

Upshot We can study Lie groups using just the linear algebra of $(\mathfrak{g}, [\cdot, \cdot])$.

Goal To classify the semisimple Lie algebras and their representations. The map $G \mapsto \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$ induces a bijection between compact connected, simply connected Lie groups and complex semisimple Lie algebras. Other motivation is given by the theory of algebraic groups, and the "ADE classification".

2 Introduction

We do some definitions, and do some basic stuff with representations, including classifying representations of $\mathfrak{sl}_2(\mathbb{C})$.

2.1 Definitions

Fix a field F, we will almost always work over $F = \mathbb{C}$ in this course. All vector space are assumed to be finite dimensional.

Definition 2.1. A *Lie algebra* is a vector space $\mathfrak{g}$ together with a bilinearing pairing known as the *Lie bracket*

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \tag{2.1}$$

such that $[\cdot, \cdot]$ is alternating, so $[x, x] = 0$ for all $x \in \mathfrak{g}$, and the Lie bracket satisfies the Jacobi identity, so that for all $x, y, z \in \mathfrak{g}$, we have that

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \tag{2.2}$$

The requirement that the Lie bracket satisfies the Jacobi identity is motivated by the adjoint representation, as we will see later.

Definition 2.2. A *Lie algebra homomorphism* is a linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ such that

$$\varphi([x, y]) = [\varphi(x), \varphi(y)] \tag{2.3}$$

for al $x, y \in \mathfrak{g}$, and φ is an isomorphism if it is a bijection.

Definition 2.3. A *Lie subalgebra* of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h} \subseteq \mathfrak{g}$ which is stable under $[\cdot, \cdot]$, so that $[x, y] \in \mathfrak{h}$ for all $x, y \in \mathfrak{h}$, so that $\mathfrak{h}$ is also a Lie algebra.

Example 2.4.

1. If $n \geq 1$ is an integer, let $\mathfrak{gl}_n(F)$ be the Lie algebra with underlying vector space $\mathrm{Mat}_n(F)$ the space of $n \times n$ matrices. The Lie bracket is given by $[A, B] = AB - BA$, we can check that this works.

2. If V is any vector space, then $\mathfrak{gl}(V) := \operatorname{End} V$ with $[f, g] = f \circ g - g \circ f$. If V is n-dimensional then choosing a basis gives an isomorphism $\mathfrak{gl}(V) \to \mathfrak{gl}_n(F)$. If $F = \mathbb{R}$, then the tangent space of $\mathrm{GL}_n(\mathbb{R})$ at 0 is $\mathfrak{gl}_n(\mathbb{R})$.

3. We have the Lie subalgebra

$$\mathfrak{sl}_n(F) = \{x \in \mathfrak{gl}_n(F) \mid \mathrm{Tr}(x) = 0\} \tag{2.4}$$

   This is a subalgebra as $\mathrm{Tr}[x, y] = \mathrm{Tr}(xy) - \mathrm{Tr}(yx) = 0$.

4. If $n = 2$, then

$$\mathfrak{sl}_2(F) = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mid a, b, c \in F \right\} \tag{2.5}$$

   has basis

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.6}$$

We can calculate that $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$. This is the most important Lie algebra, as pretty much everything we do in this course boils down to showing that things "behave like $\mathfrak{sl}_2$".

$\mathfrak{b}_n$ is the set of upper triangular matrices in $\mathfrak{gl}_n$.

$\mathfrak{n}_n$ is the set of strictly upper triangular matrices (0 in diagonal).

$\mathfrak{d}_n$ is the algebra of diagonal matrices. Note that we have $[x,y]=0$ for all $x,y \in \mathfrak{d}_n$.

Definition 2.5. A Lie algebra $\mathfrak{g}$ is *abelian* when $[x,y]=0$ for all $x,y\in\mathfrak{g}$.

There's a unique abelian Lie algebra in each dimension, which is a completely trivial fact as we just take the unique n-dimensional vector space and give it the zero Lie bracket.

Remark 2.6. Since $[x,x]=0$ we have that $[x,y]=-[y,x]$, and the converse holds when $\operatorname{char} F\neq 2$.

Given a non-degenerate bilinear form $\langle\cdot,\cdot,\rangle : V\times V\to F$, the subset

$$\mathfrak{g}=\{f\in\mathfrak{gl}_n(V)\mid \langle f(v),w\rangle+\langle v,f(w)\rangle=0\ \forall v,w\in V\} \tag{2.7}$$

is a Lie subalgebra of $\mathfrak{gl}(V)$ (exercise).

If $\langle\cdot,\cdot,\rangle$ is symmetric, then $\mathfrak{g}=\mathfrak{so}(V,\langle\cdot,\cdot,\rangle)$ is the special orthogonal Lie algebra.

If $\langle\cdot,\cdot,\rangle$ is alternating, then $\mathfrak{g}=\mathfrak{sp}(V,\langle\cdot,\cdot,\rangle)$ is the symplectic Lie algebra.

Standard $\mathfrak{so}$ and $\mathfrak{sp}$ Now if $\langle\cdot,\cdot,\rangle$ is a bilinear form, so it is nondegenerate and symmetric, and we choose a basis for V so that $V\cong F^n$, then $\langle\cdot,\cdot,\rangle$ corresponds to a symmetric matrix $A\in\operatorname{Mat}_n(F)$ such that $\langle v,w\rangle=(v^T)Aw$. Then

$$\mathfrak{so}(V,\langle\cdot,\cdot,\rangle)=\{x\in\mathfrak{gl}_n(F)\mid x^T\cdot A+A\cdot x=0\} \tag{2.8}$$

We can choose $\langle\cdot,\cdot,\rangle$ conveniently to define a "standard $\mathfrak{so}$". The matrix is given by

$$J=\begin{cases}\begin{bmatrix}0 & I_\ell\\ I_\ell & 0\end{bmatrix} & n=2\ell\\ \begin{bmatrix}1&0&0\\0&0&I_\ell\\0&I_\ell&0\end{bmatrix} & n=2\ell+1\end{cases} \tag{2.9}$$

which defines a bilinear form $\langle v,w\rangle=(v^T)Jw$ as above, and the standard $\mathfrak{so}$ is given by $\mathfrak{so}_n(F)=\mathfrak{so}(F^n,J)$.

There is a similar story for the alternating case, we set

$$J=\begin{bmatrix}0 & I_\ell\\ -I_\ell & 0\end{bmatrix} \tag{2.10}$$

and we have the standard $\mathfrak{sp}$ $\mathfrak{sp}_{2\ell}(F)=\mathfrak{sp}(F^{2\ell},J)$. Note that alternating forms only exist in even dimensional vector spaces.

Remark 2.7. Over non-closed fields bilinear forms have a more interesting structure.

Remark 2.8. You will do many matrix calculations, these are important for learning.

2.2 Representations of Lie Algebras

Let $\mathfrak{g}$ be a Lie algebra.

Definition 2.9. A *representation* of $\mathfrak{g}$ is a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V) \tag{2.11}$$

where V is a F-vector space.

Definition 2.10. A $\mathfrak{g}$-module (or $\mathfrak{g}$-action) is a vector space V and a bilinear pairing $\mathfrak{g} \times V \to V$ written as $(x, v) \mapsto x \cdot v$, such that

$$[x, y] \cdot v = x \cdot (yv) - y(x \cdot v) \tag{2.12}$$

for all $x, y \in \mathfrak{g}$ and all $v \in V$.

If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation, then $xv = \rho(x)(v)$ defines a $\mathfrak{g}$-module. Thus we have a bijection between $\mathfrak{g}$-representations and $\mathfrak{g}$-modules.

Example 2.11. If $V = F$, then $x \cdot v = 0$ for all $x \in \mathfrak{g}, v \in V$ is a $\mathfrak{g}$-module, called the *trivial representation.* This induces the map $\rho : \mathfrak{g} \to \mathfrak{gl}_1$ given by $\rho(x) = 0$ for all $x \in \mathfrak{g}$.

Example 2.12 (Defining representation)**.** If $\mathfrak{g}$ is defined as a subalgebra of $\mathfrak{gl}(V)$, then the inclusion $\mathfrak{g} \to \mathfrak{gl}(V)$ is called the defining representation.

Example 2.13. If $x \in \mathfrak{g}$, write $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ such that $\mathrm{ad}_x(y) = [x, y]$. This defines a linear map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ given by $x \mapsto \mathrm{ad}_x$.

All the nice properties of representations and linear maps come from the Lie bracket $[\cdot, \cdot]$.

Lemma 2.14. ad *is a Lie algebra homomorphism.*

Proof. We need to check that $\forall x, y \in \mathfrak{g}$, $[\mathrm{ad}_x, \mathrm{ad}_y] = \mathrm{ad}_{[x,y]}$. This follows from routine calculation. □

The representation ad is known as the *adjoint representation*, and is very important.

Example 2.15. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Then

1. $\rho_1 : \mathfrak{g} \to \mathfrak{gl}_1$ is the trivial representation given by $\rho_1(x) = 0$.
2. $\mathfrak{g} \to \mathfrak{gl}_2$ is the defining rep given by mapping e, f, h to their associated matrices.
3. $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \cong \mathfrak{gl}_3$ with basis $\{e, h, f\}$. We can calculate

$$\mathrm{ad}_e = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{ad}_h = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad \mathrm{ad}_f = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \tag{2.13}$$

2.3 Morphisms of representations

Definition 2.16. A linear map $\varphi : V \to W$ between $\mathfrak{g}$-representations is called a $\mathfrak{g}$-homomorphism (or $\mathfrak{g}$-equivariant) if it respects the representation structure on V and W, so that $\varphi(xv) = x\varphi(v)$ for all $v \in V, x \in \mathfrak{g}$.

If it is bijective, then it is a $\mathfrak{g}$-isomorphism.

Lemma 2.17. *If $\rho, \rho' : \mathfrak{g} \to \mathfrak{gl}_n(F)$ are representations, then ρ and ρ' are isomorphic if and only if there exists $M \in \mathrm{GL}_n(F)$ such that $\rho'(x) = M\rho(x)M^{-1}$ for all $x \in \mathfrak{g}$.*

Proof. An isomorphic $F^n \to F^n$ corresponds to M. □

We now list some properties of $\mathfrak{g}$-representations.

Definition 2.18. Let V be a $\mathfrak{g}$-representation, corresponding to $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. Then

1. $\dim V$ is the dimension of (or degree of) the representation.
2. V is faithful if ρ is injective.
3. A subspace $W \subseteq V$ is a subrepresentation if it is $\mathfrak{g}$-stable, so that $xw \in W$ for all $x \in \mathfrak{g}$, $w \in W$
4. V is irreducible if $V \neq 0$ and there are no non-trivial proper subrepresentations.

Example 2.19. $\mathfrak{sl}_2 \to \mathfrak{gl}_2$ the trivial representation is irreducible.
$\mathfrak{sl}_2 \to \mathfrak{gl}_2$ the defining representation is faithful and irreducible.
$\mathfrak{sl}_2 \to \mathfrak{gl}_2$ the adjoint representation is irreducible.

Lemma 2.20 (Schur's Lemma)**.** *Let V, W be irreducible $\mathfrak{g}$-representations and $\varphi : V \to W$ a $\mathfrak{g}$-homomorphism. Then*

(i) Either $\varphi = 0$, or φ is bijective (so an isomorphism).

(ii) If F is algebraically closed, and $V = W$, then $\varphi = \lambda\,\mathrm{id}_V$ for $\lambda \in F^\times$.

Proof. (i) Assume $\varphi \neq 0$. Then $\ker\varphi$ is a subrepresentation, so $\ker\varphi = 0$. $\operatorname{im}\varphi$ is also a subrepresentation, so $\operatorname{im}\varphi = W$.

(ii) Since F is algebraically closed, φ has an eigenvalue $\lambda \in F$. Then $\varphi - \lambda\,\mathrm{id}_V$ has an eigenvector in its kernel, so $\varphi - \lambda\,\mathrm{id}_V$ is not bijective, so $\varphi - \lambda\,\mathrm{id}_V = 0$.

□

2.4 Representations of $\mathfrak{sl}_2$

Our goal is to classify all irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$. Let V be an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$, with $\rho : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$. For example, the trivial, defining, or adjoint representations. For $\lambda \in \mathbb{C}$, we can define the eigenspace

$$V_\lambda = \{v \in V \mid hv = \lambda v\} \tag{2.14}$$

We want to understand how e and f interact with V_λ.

Lemma 2.21. *We have that $e(V_\lambda) \subset V_{\lambda+2}$ and $f(V_\lambda) \subset V_{\lambda-2}$.*

Proof. If $v \in V_\lambda$ so that $hv = \lambda v$, then direct calculation gives the result. $\square$

So applying e and f allows us to "hop" through the eigenspaces.

Since h (or really $\rho(h)$) has at least one eigenvalue, there exists $\lambda \in \mathbb{C}$ such that $V_\lambda \neq 0$. Then the lemma shows that

$$\bigoplus_{k\in\mathbb{Z}} V_{\lambda+2k} \tag{2.15}$$

is stable under the action of h, e, f, so it must equal the entire space V because ρ is an irreducible representation. So

$$\bigoplus_{k\in\mathbb{Z}} V_{\lambda+2k} \tag{2.16}$$

where $V_\lambda \neq 0$. At some point, we must have that $V_{\lambda+2k} = 0$.

Definition 2.22. If $\lambda \in \mathbb{C}$ is such that $V_\lambda \neq 0$, then λ is a *weight* of V.

If $V_\lambda \neq 0$ but $V_{\lambda+2} = 0$, then λ is a *highest weight*, and $v \in V_\lambda$ is a *highest weight vector.*

Lemma 2.23. *If $v \in V_\lambda$ is a highest weight vector, then $W = \langle v, fv, f^2v, \cdots \rangle$ is a subrepresentation of V.*

Proof. We need to check that W is stable under the action of e, h, f. We can do this by direct calculation. Importantly, we check that

$$e(f^k v) = k(\lambda - k + 1)(f^{k-1}v) \tag{2.17}$$

$\square$

So if v is a highest weight vector, then $V = \langle v, fv, f^2v, \ldots \rangle$ because W is a nonzero subrepresentation and V is an irreducible representation.

Corollary 2.24. *All the weight spaces are one-dimensional spanned by $f^k v$, where v is a highest weight vector and $k \geq 0$.*

Corollary 2.25. *If V has dimension $n+1$, then V has highest weight n.*

Proof. Let $v \in V_\lambda$ be a highest weight vector. Then $V = \{v, fv, \ldots, f^n v\}$ because $\dim V = n+1$ and $f^k v \neq 0$ for $0 \leq k \leq n$. Now by (2.17) we have that $e \cdot (f^{n+1}v) = (n+1)(\lambda - n)(f^n v) = 0$ so $\lambda = n$. $\square$

So the weights of V are $\{-n, -n+1, \ldots, n-2, n\}$.

Theorem 2.26. *For every $n \in \mathbb{Z}_{\geq 0}$, there is a unique isomorphism class of irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ of dimension $n+1$, denoted by $V(n)$.*

Proof. For uniqueness, if V is an irreducible representation of dimension $n+1$ and $v \in V$ is a highest weight vector, then we have a basis $\{v, fv, \ldots, f^n v\}$ for V, and this determines all the matrices for the representation ρ.

For existence, one can check that the matrices which are defined preserve the bracket. For irreducibility, if W is a nonzero subrepresentation of $V(n)$, and $v_0, \ldots, v_n$ is the standard basis of $V(n)$, then there exists $v = \sum c_i v_i \in W \setminus \{0\}$. Since $ev_{i+1} = av_i$, we have that $e^k v = av_0$ with $a \neq 0$ for some k, so $v_0 \in W$. Then $f^j v_0 = bv_j$, so $v_j \in W$ for all j, so $W = V$. $\square$

2.5 Complete reducibility

Let $\mathfrak{g}$ be any Lie algebra over any field F.

Definition 2.27. If V, W are $\mathfrak{g}$-representations, then $V \oplus W$ can be given the structure of a $\mathfrak{g}$-representation via $\mathfrak{g}(v, w) = (\mathfrak{g}v, \mathfrak{g}w)$. A $\mathfrak{g}$-representation is *completely reducible* if $V \cong V_1 \oplus \cdots \oplus V_n$ where each V_i is irreducible.

Lemma 2.28. *A representation V is completely reducible if and only if for every subrepresentation $W \subset V$, there exists a subrepresentation $W' \subset V$ such that $W \oplus W' = V$.*

The above lemma says that a representation is completely reducible if and only if the complement of every subrepresentation is a subrepresentation.

Proof. First suppose every complement is a subrepresentation. If we induct on dimension, then our decomposition will eventually stop.

Now suppose V is completely reducible and W is a subrepresentation. Suppose W' is a subrepresentation such that $W \cap W' = \{0\}$ and W' is maximal among all subrepresentations with this property. Then $V = W \oplus W'$ (exercise). $\square$

Example 2.29. Let $\mathfrak{g} = \mathfrak{b}_2 \subset \mathfrak{gl}_2$ the Lie algebra of upper triangular matrices. Then the defining representation is not completely reducible. This is because $\mathfrak{b}_2$ preserves the first basis vector e_1, so $W = \langle e_1 \rangle$ is a subrepresentation, but its complement $W' = \langle e_2 \rangle$ is not a subrepresentation.

2.6 More operations of representations

Definition 2.30. If $W \subset V$ is a subrepresentation of a $\mathfrak{g}$-representation, then $V/W = \{v + W \mid v \in V\}$ is a $\mathfrak{g}$-representation because W is stable under the $\mathfrak{g}$-action. We have the obvious action

$$x(v + W) = xv + W. \tag{2.18}$$

Definition 2.31. If V is a $\mathfrak{g}$-representation, then the dual space $V^* = \mathrm{Hom}(V, F)$ is a $\mathfrak{g}$-representation with action given by

$$(xf)(v) = -f(xv) \tag{2.19}$$

for $x \in \mathfrak{g}$, $f \in V^*$, $v \in V$.

Definition 2.32. If V, W are $\mathfrak{g}$-representations, then $\mathrm{Hom}_F(V, W)$ is a $\mathfrak{g}$-representation via

$$(x\varphi)(v) = x\varphi(v) - \varphi(xv) \tag{2.20}$$

for $\varphi \in \mathrm{Hom}_F(V, W)$, $x \in \mathfrak{g}$, $v \in V$.

Definition 2.33. If V, W are $\mathfrak{g}$-representations, then $V \otimes_F W$ is a $\mathfrak{g}$-representation by $x(v \otimes w) = (xv) \otimes w + v \otimes (xw)$.

Definition 2.34. We have that $\mathrm{Sym}^n V, \bigwedge^n V$ are subrepresentations of $V^{\otimes n}$.

2.7 Complete reducibility for $\mathfrak{sl}_2(\mathbb{C})$

If V is an $\mathfrak{sl}_2(\mathbb{C})$-representation, then the set $\{\lambda \in V \mid V_\lambda \neq 0\}$ is called the set of weights. We view this as a multiset where λ has multiplicity $\dim V_\lambda$, this is reflected in the multiple roots of the characteristic polynomial.

As a consequence of complete reducibility and our description of $V(n)$, we have the following.

Corollary 2.35. *A representation of $\mathfrak{sl}_2(\mathbb{C})$ is determined up to isomorphism by its weights (with multiplicities).*

Proof. Trivial $\square$

Example 2.36.

1. Suppose an $\mathfrak{sl}_2$ representation V has weights $\{5, 3, 3, 1, 1, 0, -1, -1, -3, -3, -5\}$. Then $V = V(0) \oplus V(3) \oplus V(5)$.
2. Suppose V has dimension 5 and has 3 as a weight. Then $V = V(3) \oplus V(0)$.
3. The defining representation if $V(1)$.
4. The adjoint representation is $V(2)$.

Definition 2.37. Let $\rho : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$ be a representation. The *Casimir element* is the map

$$\Omega_\rho = \rho(e)\rho(f) + \rho(f)\rho(e) + \frac{1}{2}\rho(h)^2 \in \mathfrak{gl}(V) = \mathrm{End}(V) \tag{2.21}$$

We consider $\mathrm{End}(V)$ as a ring, but ρ is not a ring homomorphism. For instance, we do not necessarily have $\rho(e)\rho(f) = \rho(ef)$ because "ef" is a meaningless concept in a Lie algebra.

Lemma 2.38. *$\Omega_\rho : V \to V$ is $\mathfrak{g}$-equivariant, so it gives a homomorphism of representations.*

Proof. We need to show that $\Omega_\rho(xv) = x\Omega_\rho(v)$ for all $x \in \mathfrak{sl}_2(\mathbb{C})$ and $v \in V$. Equivalently, we want to show that $\rho(x)\Omega_\rho = \Omega_\rho\rho(x)$ for $x = h, e, f$. We can do this by direct calculation. $\square$

Lemma 2.39. *If $\rho \cong V(n)$, then $\Omega_\rho = c\,\mathrm{id}$ where $c = (n^2/2 + n)$*

Proof. By Schur's lemma, Ω_ρ defines a subrepresentation (taking images) so $\Omega_\rho = c\,\mathrm{id}$. Let $v \in V(n)$ be a highest weight vector. Then $hv = nv$, and $ev = 0$. So $\Omega_\rho v = (n^2/2 + n)v$ which is nonzero when $n > 0$, so when ρ is nontrivial. $\square$

Lemma 2.40. *If $\mathfrak{h}$ is an abelian Lie algebra, then every Lie algebra homomorphism $\varphi : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{h}$ is 0.*

Proof. We have that $\varphi([x, y]) = [\varphi(x), \varphi(y)] = 0$. Since $[x, y]$ spans $\mathfrak{sl}_2(\mathbb{C})$ so that $[\mathfrak{sl}_2(\mathbb{C}), \mathfrak{sl}_2(\mathbb{C})] = \mathfrak{sl}_2(\mathbb{C})$, we have that $\varphi(x) = 0$ for all x. $\square$

To show that an $\mathfrak{sl}_2(\mathbb{C})$ representation V is completely reducible, we'll show that every subrepresentation $W \subseteq V$ has an invariant complement $W' \subseteq V$ such that $V = W \oplus W'$ (and $W \cap W' = 0$).

Proposition 2.41. *If $W \subseteq V$ has codimension 1, it has a complementary representation, so there exists a trivial subrepresentation $W' \subseteq V$ such that $W \oplus W' = V$.*

Proof. $V/W \cong \mathbb{C}$ is an $\mathfrak{sl}_2(\mathbb{C})$ representation, and since $\mathfrak{gl}(V/W) \cong \mathfrak{gl}_1$ is abelian, V/W is trivial. Choose a basis $e_1, \ldots, e_n$ for V such that $e_1, \ldots, e_{n-1}$ is a basis for W. In this basis, $\rho : \mathfrak{sl}_2 \to \mathfrak{gl}(V)$ looks like

$$\left(\begin{array}{ccc|c} & & & * \\ & \rho|_W & & * \\ & & & * \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \tag{2.22}$$

We will choose e_n so that $\rho(x)(e_n) = 0$ for all $x \in \mathfrak{sl}_2$.

Case 1: If $W \cong V(0)$ is trivial, then $\operatorname{im}\rho$ is abelian, so $\rho = 0$.

Case 2: Assume W is irreducible and nontrivial. Consider $\Omega_\rho : V \to V$. We have that $\Omega(W) \subset W$ because W is a subrepresentation. W is irreducible, so $\Omega|_W = c\,\mathrm{id}_W$ for $c \neq 0$ by Lemma 1. On the other hand, $\overline{\Omega} : V/W \to V/W$ is zero by Lemma 2.39. Therefore $W' = \ker \Omega$ has the property that $V = W \oplus W'$, and it is a subrepresentation because of equivariance (the kernel of a $\mathfrak{g}$-equivariant homomorphism is a representation).

Case 3: Now let W be any subrepresentation. We proceed by induction on $\dim W$. If $\dim W = 0$ or W is irreducible, we are done by the previous cases. So we can assume that W is nonzero and reducible. Let $U \subsetneq W$ be a proper, nonzero subrepresentation. Then W/U is a subrepresentation of V/U of codimension 1, and W/U has dimension strictly smaller than W. By the induction hypothesis, there exists a one-dimensional subrepresentation $L \subset V/U$ such that $V/U = (W/U) \oplus L$. So $L = W'/U$ for some $W' \subset V$ containing U. Then $V = W + W'$ and $W \cap W' = U$. Since L is one-dimensional, $U \subset W'$ has codimension 1. By the induction hypothesis, there exists $W'' \subseteq W'$ such that $W' = U \oplus W''$. Therefore $V = W \oplus W''$.

□

Theorem 2.42. *Let V be a representation of $\mathfrak{sl}_2$ and W a subrep. Then there exists W' such that $V = W \oplus W'$.*

Proof. Recall $\operatorname{Hom}(V, W)$ is an $\mathfrak{sl}_2$ rep via $(x\varphi)(v) = x(\varphi(v)) - \varphi(xv)$. Let

$$\mathcal{V} := \{\varphi \in \operatorname{Hom}(V, W) \mid \varphi|_W = c\,\mathrm{id}_W\} \supset \mathcal{W} := \{\varphi \in \operatorname{Hom}(V, W) \mid \varphi|_W = 0\} \tag{2.23}$$

We claim that $\mathcal{W}$ and $\mathcal{V}$ are subrepresentations of $\operatorname{Hom}(V, W)$ and $\mathcal{W} \subset \mathcal{V}$ has codimension 1. Indeed, if $\varphi \in \mathcal{V}$ and $x \in \mathfrak{sl}_2(\mathbb{C})$, $w \in W$, then $(x\varphi)(w) = x\varphi(w) - \varphi(xw) = 0$ since $\varphi|_W = c\,\mathrm{id}_W$. and W is a subrepresentation of V. Thus $x\mathcal{V} \subset W$, so $\mathcal{V}$ and $\mathcal{W}$ are subrepresentations. The map $\mathcal{F} : \mathcal{V} \to \mathbb{C}$ sending φ to c, where $\varphi|_W = c\,\mathrm{id}_W$ is linear and surjective. Thus $\ker \mathcal{F} = \mathcal{W}$ has codimension 1.

By the proposition, there exists a trivial subrepresentation $\mathcal{W}' \subset \mathcal{V}$ such that $\mathcal{V} = \mathcal{W} \oplus \mathcal{W}'$. Let $\varphi \in \mathcal{W}'$ be nonzero. Now, $x\varphi = 0$ for all $x \in \mathfrak{sl}_2(\mathbb{C})$ since $\mathcal{W}'$ is trivial. Thus φ is $\mathfrak{sl}_2(\mathbb{C})$ equivariant as $(x\varphi)(v) = x\varphi(v) - \varphi(xv) = 0$ for all $v \in V$. Thus $W' = \ker\varphi$ satisfies $V = W \oplus W'$. □

2.8 Tensor products

Let F be a field. Recall the tensor product. Recall that not all tensors are pure.

Lemma 2.43 (Universal property of tensor product)**.** *If V, W, U are vector spaces, then there is a canonical isomorphism between* $\operatorname{Hom}(V \otimes W, U)$ *and the set of bilinear maps* $V \times W \to U$ *given by* $(v \otimes w \mapsto u) \mapsto ((v, w) \mapsto u)$.

Example 2.44. In $F^2 \otimes F^3$, the elements are the form $\sum c_{ij} e_i \otimes f_j$, and we can represent this as a matrix. In this way pure tensors are rank 1 matrices.

If $\mathfrak{g}$ is a Lie algebra and V, W are $\mathfrak{g}$-representations, then $V \otimes W$ is a representation via $x(v \otimes w) = (xv) \otimes w + v \otimes (xw)$ for all $x \in \mathfrak{g}$, $v \in V$, $w \in W$. Notice the similarities between this and the product rule.

Definition 2.45. If $n \geq 1$ and V is a vector space, let $V^{\otimes n} = V \otimes \cdots \otimes V$ be the nth tensor power.

Let

$$\operatorname{Sym}^n V = V^{\otimes n} / \operatorname{Span}\{v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \mid \sigma \in S_n\} \tag{2.24}$$

be the nth symmetric power, spanned by the symbols $v_1 \cdots v_n$ where we can change the order of the v_is as we wish. Also, let

$$\bigwedge^n V = V^{\otimes n} / \operatorname{Span}\{v_1 \otimes \cdots \otimes v_n - \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \mid \sigma \in S_n\} \tag{2.25}$$

be the nth exterior power, spanned by the symbols $v_1 \wedge \cdots \wedge v_n$ with $v_i \neq v_j$.

If V is a $\mathfrak{g}$-representation, then the tensor, symmetric, and exterior powers are also.

Example 2.46. If V has basis $e_1, \ldots, e_n$, then

1. $V^{\otimes 2}$ has basis $\{e_i \otimes e_j \mid 1 \leq i, j \leq n\}$.
2. $\operatorname{Sym}^2 V$ has basis $\{e_i e_j \mid 1 \leq i \leq j \leq n\}$.
3. $\bigwedge^2 V$ has basis $\{e_i \wedge e_j \mid 1 \leq i < j \leq n\}$.

3 Solvability and Nilpotents

We define what it means for Lie algebras and elements of Lie algebras to be solvable, nilpotent.

3.1 Ideals of Lie Algebras

Let $\mathfrak{g}$ be a Lie algebra over F.

Definition 3.1. An *ideal* of $\mathfrak{g}$ is a subspace $I \subset \mathfrak{g}$ such that for all $x \in \mathfrak{g}$, $[x, I] \subset I$. This is always a subalgebra because $[I, I] \subset [\mathfrak{g}, I] \subset I$.

Remark 3.2. In analogy with Lie groups we have

$$\begin{aligned} \text{Lie algebra} &\leftrightarrow \text{Lie group} \\ \text{subalgebra} &\leftrightarrow \text{subgroup} \\ \text{ideal} &\leftrightarrow \text{normal subgroup} \end{aligned}$$

Remark 3.3. An alternative definition of an ideal is a subrepresentation of $\operatorname{Ad} \mathfrak{g}$.

Lemma 3.4. *If $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, then* $\ker \varphi$ *is always an ideal.*

Proof. If $x \in \mathfrak{g}$, $y \in \ker \varphi$, then $\varphi([x,y]) = [\varphi(x), \varphi(y)] = 0$ so $[x,y] \in \ker \varphi$. ☐

Lemma 3.5. *If $I \subset \mathfrak{g}$ is an ideal, then $\mathfrak{g}/I$ has the structure of a Lie algebra via $[x+I, y+I] = [x,y] + I$.*

The projection $\mathfrak{g} \to \mathfrak{g}/I$ is a homomorphism with kernel I.

Proof. Trivial. ☐

Definition 3.6. A Lie algebra $\mathfrak{g}$ is *simple* if it has no nonzero proper ideals and $\mathfrak{g}$ is not abelian.

Remark 3.7. $\mathfrak{g}$ is simple if and only if $\mathfrak{g}$ is not abelian and the adjoint representation is irreducible.

Example 3.8. $\mathfrak{sl}_2(\mathbb{C})$ is simple since the adjoint representation $V(2)$ is irreducible.

Example 3.9. $\mathfrak{sl}_n(\mathbb{C}), \mathfrak{sp}_{2\ell}(\mathbb{C}), \mathfrak{so}_n(\mathbb{C})$ are all simple.

We'll classify all simple Lie algebras over $\mathbb{C}$, and we'll find that the the three listed above are almost all the examples, except for 5 "exceptional" Lie algebras.

3.2 Solvable and Nilpotent Lie Algebras

If $I, J \subset \mathfrak{g}$ are ideal, let $[I,J] = \operatorname{Span}\{[i,j] \mid i \in I, j \in J\}$. Note that we need to take the span as simply taking $[i,j]$ for all $i \in I$ and $j \in J$ might not give a subspace. Then $[I,J]$ is an ideal, which follows from the Jacobi identity.

Definition 3.10. The *derived subalgebra* of $\mathfrak{g}$ is $[\mathfrak{g}, \mathfrak{g}]$.

Remark 3.11. $[\mathfrak{g}, \mathfrak{g}] = 0$ if and only if $\mathfrak{g}$ is abelian.

Example 3.12. $[\mathfrak{gl}_n, \mathfrak{gl}_n] = \mathfrak{sl}_n$.

Example 3.13. If $\mathfrak{g}$ is simple, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ because $[\mathfrak{g}, \mathfrak{g}]$ is a nonzero ideal (recall that it is nonzero because a simple Lie algebra must not be abelian).

Definition 3.14. Set $\mathfrak{g}^{(0)} = \mathfrak{g}^0 = \mathfrak{g}$, and $\mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}]$ and $\mathfrak{g}^n = [\mathfrak{g}, \mathfrak{g}^{n-1}]$. We then have filtrations of Lie algebras

$$\mathfrak{g} \supset \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \cdots \qquad \text{the } \textit{derived series} \tag{3.1}$$

$$\mathfrak{g} \supset \mathfrak{g}^{1} \supset \mathfrak{g}^{2} \supset \cdots \qquad \text{the } \textit{central series} \tag{3.2}$$

$\mathfrak{g}$ is *nilpotent* if $\mathfrak{g}^n = 0$ for some $n \geq 1$. $\mathfrak{g}$ is *solvable* if $\mathfrak{g}^{(n)} = 0$ for some $n \geq 1$.

Remark 3.15. We have that $\mathfrak{g}^{(n)} \subset \mathfrak{g}^n$, so nilpotent implies solvable.

Example 3.16.

1. $\mathfrak{n}_n$ the algebra of strictly upper triangular matrices is nilpotent.
2. $\mathfrak{b}_n$ the algebra of upper triangular matrices is solvable but not nilpotent for $n \geq 2$.

Lemma 3.17. *If $\mathfrak{g} = \mathfrak{n}_n(F)$, then $\mathfrak{g}^n = 0$, so $\mathfrak{g}$ is nilpotent.*

Proof. An $n \times n$ matrix has *level* greater than k if the nonzero entries of A are supported on indices with $j - i \geq k$. Basically it measures how upper triangular the matrix is. Each term in the central series of $\mathfrak{n}_n$ increases the level by 1. $\square$

Remark 3.18. A similar argument shows that $(\mathfrak{b}_n)^{(m)}$ are the matrices with level m. But $\mathfrak{b}_n^m = \mathfrak{n}_n$, so $\mathfrak{b}_n$ is not nilpotent.

Lemma 3.19. *If $\mathfrak{g}$ is nilpotent (or solvable), then so is any (i) subalgebra or (ii) quotient.*

Proof. (i) If $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra, then $\mathfrak{h}^{(n)} \subset \mathfrak{g}^{(n)}$ and $\mathfrak{h}^n \subset \mathfrak{g}^n$.

(ii) If $I \subset \mathfrak{g}$ is an ideal, then $(\mathfrak{g}/I)^{(n)} = \mathfrak{g}^{(n)} + I$ and $(\mathfrak{g}/I)^n = \mathfrak{g}^n + I$. $\square$

3.3 Engel's Theorem

Our arguments still work over general fields.

Recall that an endomorphism $\varphi \in \mathfrak{gl}(V)$ is nilpotent if $\varphi^n = 0$ for some $n \geq 1$. Equivalently, all the eigenvalues of φ over an algebraic closure are 0.

Lemma 3.20. *If $\mathfrak{g}$ is nilpotent, then for all $x \in \mathfrak{g}$, $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ is nilpotent.*

Proof. $\mathfrak{g}$ is nilpotent if and only if there exists $n \geq 1$ such that $\mathrm{ad}_{x_1} \circ \cdots \circ \mathrm{ad}_{x_n} = 0$ for all $x_1, \ldots, x_n$. In particular, taking $x = x_1 = \cdots = x_n$, we have that $\mathrm{ad}^n x = 0$. $\square$

Theorem 3.21 (Engel)**.** *$\mathfrak{g}$ is nilpotent if and only if ad_x is nilpotent for all $x \in \mathfrak{g}$.*

One direction of this equivalence follows from Lemma 3.20.

Remark 3.22. It is not true that $\mathfrak{g} \subset \mathfrak{gl}_n$ is nilpotent if and only if x is a nilpotent matrix for all $x \in \mathfrak{g}$.

Lemma 3.23. *If V is a vector space and $x \in \mathfrak{gl}(V)$ is nilpotent, then $\mathrm{ad}_x \in \mathfrak{gl}(\mathfrak{gl}(V))$ is nilpotent.*

Proof. We have that $\mathrm{ad}_x(y) = xy - yx = L_x(y) - R_x(y)$ where L_x and R_x are multiplication on the left and right. Since x is nilpotent, L_x and R_x are nilpotent operators. Also, $L_x \circ R_x = R_x \circ L_x$, so ad_x is nilpotent (iterated applications of ad_x are sums of iterated applications of L_x, R_x, which will be zero of large n). $\square$

Proposition 3.24. *If $\mathfrak{g} \in \mathfrak{gl}(V)$ is a subalgebra such that x is nilpotent for all $x \in \mathfrak{g}$, then there is a nonzero $v \in V$ such that $xv = 0$ for all $x \in \mathfrak{g}$.*

Proof. We proceed by induction of $\dim \mathfrak{g}$. If $\dim \mathfrak{g} = 1$, then $\mathfrak{g} = \mathrm{Span}\{x\}$ with x nilpotent, so $x^n = 0$ for some minimal n, so there exists v such that $x(x^{n-1}v) = 0$ and $x^{n-1}v \neq 0$.

For the inductive step, let $\mathfrak{p} \subset \mathfrak{g}$ be a maximal proper subalgebra. If $x \in \mathfrak{p}$, then $\mathrm{ad}(x)$ is nilpotent by Lemma 3.23. So $\overline{ad(x)} : \mathfrak{g}/\mathfrak{p} \to \mathfrak{g}/\mathfrak{p}$ is a nilpotent element of $\mathfrak{gl}(\mathfrak{g}/\mathfrak{p})$, and in this way we can consider $\mathfrak{p}$ as a subalgebra of $\mathfrak{gl}(\mathfrak{g}/\mathfrak{p})$ with each element nilpotent. Then by the inductive hypothesis, there exists an element $\overline{y} \in \mathfrak{g}/\mathfrak{p}$ such that $\overline{\mathrm{ad}(x)}(\overline{y}) = 0$ for all $x \in \mathfrak{p}$. Lifting $\overline{y}$ to some $y \in \mathfrak{g}$, we have that $y \notin \mathfrak{p}$ and $[x, y] \in \mathfrak{p}$ for all $x \in \mathfrak{p}$. Therefore $\mathfrak{p} + F \cdot y$ is a subalgebra, so $\mathfrak{g} = \mathfrak{p} + F \cdot y$ since we chose $\mathfrak{p}$ to be maximal and $y \notin \mathfrak{p}$.

Moreover, $\mathfrak{p}$ is an ideal of $\mathfrak{g}$, since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, y] \subset \mathfrak{p}$. Consider

$$W = \{v \in V \mid pv = 0 \forall p \in \mathfrak{p}\} \tag{3.3}$$

By the induction hypothesis applied to $(\mathfrak{p}, V)$, $W \neq 0$. We claim that W is stable under y, so that $y \cdot W \subset W$. Indeed, if $v \in W$ and $p \in \mathfrak{p}$, then we need to check that $yv \in W$:

$$p(yv) = [p, y]v + y(pv) = 0 \tag{3.4}$$

because $[p, y] \in \mathfrak{p}$. Now, by the dimension 1 case applied to $\operatorname{Span}\{y\} \subset \mathfrak{gl}(W)$, there exists $w \in W \setminus \{0\}$ such that $yw = 0$. Then w satisfies the statement of the theorem.

□

Corollary 3.25. *If $\mathfrak{g} \in \mathfrak{gl}(V)$ is a subalgebra such that x is nilpotent for all $x \in \mathfrak{g}$, then there exists a basis of V such that $\mathfrak{g} \subset \mathfrak{n}_n$, so $\mathfrak{g}$ is a nilpotent Lie algebra.*

Proof. By Proposition 3.24, there exists $v_1 \in V \setminus \{0\}$ such that $\mathfrak{g} \cdot v_1 = 0$. Choosing a basis $v_1, \ldots, v_n$, we have that the first column of the matrix for each element of $\mathfrak{g}$ is all zeros. Applying the proposition to $V/\operatorname{Span}\{v_1\}$ and iterating, we find that $\mathfrak{g} \subset \mathfrak{n}_n$ is upper triangular. □

Proof of Theorem 3.21. We need to show that if ad_x is nilpotent for all $x \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent. The other direction is done by Lemma 3.20.

Consider $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ and $\mathfrak{h} = \operatorname{im} \mathrm{ad}$. By the Corollary, $\mathfrak{h}$ is nilpotent. Since $\ker \mathrm{ad} = Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0 \forall y \in \mathfrak{g}\}$, we have that $\mathfrak{h} = \mathfrak{g}/Z(\mathfrak{g})$, so $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent. So $(\mathfrak{g}/Z(\mathfrak{g}))^n = 0$ for some $n \geq 1$. So $\mathfrak{g}^n + Z(\mathfrak{g}) = Z(\mathfrak{g})$, so $\mathfrak{g}^n \in Z(\mathfrak{g})$, so $\mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^n] \subset [\mathfrak{g}, Z(\mathfrak{g})] = 0$. □

From now on, we will assume that $F = \mathbb{C}$.

3.4 The big three theorems

The first big theorem is Engel's Theorem 3.21, which we have already proved. We proved it by showing if a subalgebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ consists of nilpotent elements, there is a basis of V such that $\mathfrak{g} \subset \mathfrak{n}_n$.

The second big theorem is Lie's theorem, which is an analogue of Engel's theorem for solvable subalgebras. We do not prove it, but you can see Humphreys.

Theorem 3.26 (Lie)**.** *If $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a solvable subalgebra, then there exists a basis of V such that $\mathfrak{g} \subset \mathfrak{b}_n$.*

The third big theorem gives a trace criterion for solvability. We also omit the proof.

Theorem 3.27 (Cartan)**.** *A subalgebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ is solvable if and only if for all $x \in \mathfrak{g}$, and for all $y \in [\mathfrak{g}, \mathfrak{g}]$, $\operatorname{Tr}(xy) = 0$.*

Proof. If $\mathfrak{g} \subset \mathfrak{gl}(V)$ is solvable, by Lie's theorem we can pick a basis of V such that $\mathfrak{g} \subset \mathfrak{b}_n$. If $x \in \mathfrak{b}_n$ and $y \in [\mathfrak{b}_n, \mathfrak{b}_n] = \mathfrak{n}_n$, then $xy \in \mathfrak{n}_n$, so $\operatorname{Tr}(xy) = 0$.

For the other direction, see Humphreys, §4.3. □

4 Semisimpleness is very nice!

We define semisimpleness.

We also define the trace form, and use it to show representations of semisimple Lie algebras are completely reducible.

Also we do Jordan decomposition for elements of semisimple Lie algebras.

4.1 Semisimple Lie Algebras

Definition 4.1. A Lite algebra $\mathfrak{g}$ is *semisimple* if $\mathfrak{g} = I_1 \oplus I_2 \oplus \cdots \oplus I_k$ where each I_i is a simple Lie algebra.

Example 4.2.

1. Every simple Lie algebra is semisimple.
2. We have that $\mathfrak{so}_4(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ is simple but not semisimple.

Lemma 4.3. *A Lie algebra $\mathfrak{g}$ is semisimple if and only if $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is faithful and completely reducible.*

Proof. $\mathfrak{g}$ has no abelian ideals if and only if $Z(\mathfrak{g})$ is trivial if and only if ad is faithful.

$\mathfrak{g}$ has a decomposition into irreducible ideals if and only if ad is completely reducible. □

4.2 The Killing form

Definition 4.4. If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$, the *trace form* is the bilinear form

$$\begin{aligned} (\cdot, \cdot)_V : \mathfrak{g} \times \mathfrak{g} &\to \mathbb{C} \\ (x, y)_V &= \mathrm{Tr}_V(\rho(x)\rho(y)) \end{aligned} \tag{4.1}$$

Recall that trace is basis independent.

Definition 4.5. The *Killing form* is the trace form of the adjoint representation:

$$\kappa(x, y) \coloneqq \mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}(x)\,\mathrm{ad}(y)). \tag{4.2}$$

This is clearly symmetric and bilinear.

Lemma 4.6. *If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation, then $([x, y], z)_V = (x, [y, z])_V$ for all $x, y, z \in \mathfrak{g}$.*

Proof. Direct calculation: use trace commutativity. □

Warning! $\mathrm{Tr}(ABC) \neq \mathrm{Tr}(BAC)$ in general.

Example 4.7. Let $\mathfrak{g} = \mathfrak{sl}_2$. In the basis $\{e, h, f\}$ we can calculate $\kappa(e, e) = 0$, $\kappa(h, h) = 8$, and $\kappa(f, f) = 0$. The Gram matrix of κ is

$$\begin{bmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{bmatrix} \tag{4.3}$$

The determinant of this matrix is nonzero, so κ is nondegenerate. In fact, this characterizes semisimple Lie algebras.

4.3 The Cartan-Killing criterion

Lemma 4.8.

(i) If $I \subset \mathfrak{g}$ is an ideal such that I and $\mathfrak{g}/I$ are solvable, then $\mathfrak{g}$ is solvable.

(ii) If I, J are two solvable ideals, then so is $I + J$.

Proof. (i) If I and $\mathfrak{g}/I$ are solvable, then there exists $n \geq 1$ such that $(\mathfrak{g}/I)^{(n)} = \mathfrak{g}^{(n)} + I = 0 + I$ so $\mathfrak{g}^{(n)} \subset I$. But there exists $m \geq 1$ such that $I^{(m)} = 0$, so $\mathfrak{g}^{(nm)} \subset I^{(m)} = 0$.

(ii) $I + J$ is isomorphic to $(I \oplus J)/(I \cap J)$. Since I and J are solvable, so is $I \oplus J$ and hence $I + J$. □

Definition 4.9. By the previous lemma, any two solvable ideals are contained in their sum, so we can take the sum of all of them to get a unique maximal ideal solvable ideal. This is called the *radical* of $\mathfrak{g}$, and is denoted by $\mathrm{Rad}(\mathfrak{g})$.

Theorem 4.10 (Cartan-Killing criterion)**.** *Let $\mathfrak{g}$ be a nonzero Lie algebra over $\mathbb{C}$. The following are equivalent:*

(i) *$\mathfrak{g}$ is semisimple.*

(ii) $\mathrm{Rad}(\mathfrak{g}) = 0$.

(iii) *The Killing form is nondegenerate.*

Before we prove this, we need a little lemma.

Lemma 4.11. *If $I \subset \mathfrak{g}$ is an ideal and κ_I is the Killing form of I, then $\kappa(x, y) = \kappa_I(x, y)$ for all $x, y \in I$.*

Proof. Consider two adjoint representations: $\mathrm{ad}_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ and $\mathrm{ad}_I : I \to \mathfrak{gl}(I)$. If $x \in I$, then $\mathrm{ad}_{\mathfrak{g}}(x)$ sends $\mathfrak{g}$ to I, so the matrix looks like

$$\mathrm{ad}_{\mathfrak{g}}(x) = \begin{pmatrix} \mathrm{ad}_I(x) & * \\ 0 & 0 \end{pmatrix} \tag{4.4}$$

so $\mathrm{Tr}(\mathrm{ad}_{\mathfrak{g}}(x)\,\mathrm{ad}_{\mathfrak{g}}(y)) = \mathrm{Tr}(\mathrm{ad}_I(x)\,\mathrm{ad}_I(y))$. □

Proof of Theorem 4.10.

(i) $\to$ (ii):

Write $\mathfrak{g} = I_1 \oplus \cdots \oplus I_k$ with I_i simple. Let R_j be the projection of $\mathrm{Rad}(\mathfrak{g})$ onto $\mathfrak{g}/I_j$. Since R_j is the quotient of a solvable Lie algebra $\mathrm{Rad}(\mathfrak{g})$, R_j is solvable. Also, R_j is an ideal of I_j because projection is surjective.

Since I_j is simple, every ideal of I_j is 0 or I_j. Also, since $[I_j, I_j] = I_j$ because I_j is simple, I_j is not solvable,so $R_j = 0$ for all j. So $\mathrm{Rad}(\mathfrak{g}) = 0$.

(ii) $\to$ (iii):

Set

$$\mathfrak{g}^{\perp} = \{x \in \mathfrak{g} \mid \kappa(x, y) = 0 \forall y \in \mathfrak{g}\}. \tag{4.5}$$

We want to show that $\mathfrak{g}^{\perp}$ is solvable, so if $\mathrm{Rad}(\mathfrak{g}) = 0$, then κ is nondegenerate.

First, $\mathfrak{g}^{\perp}$ is a subspace because it is the kernel of a bilinear form. It is an ideal because $\kappa([x, y], z) = \kappa(x, [y, z])$, so if $x \in \mathfrak{g}^{\perp}$ and $y \in \mathfrak{g}$, then for all $z \in \mathfrak{g}$,

$$\kappa([y, x], z) = -\kappa(x, [y, z]) = 0 \tag{4.6}$$

so $[y, z] \in \mathfrak{g}^{\perp}$. So $[\mathfrak{g}, \mathfrak{g}^{\perp}] \subset \mathfrak{g}^{\perp}$ so it is an ideal.

To show that $\mathfrak{g}^{\perp}$ is solvable, we use Cartan's trace Theorem 3.27. So apply this, consider $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ and consider the subalgebra $\mathfrak{h} = \mathrm{ad}(\mathfrak{g}^{\perp})$. By Cartan's theorem, this is solvable because $\mathrm{Tr}(xy) = 0$ for all $x, y \in \mathfrak{h}$. Moreover, $\mathfrak{h} \cong \mathfrak{g}^\perp / Z(\mathfrak{g})$, so $\mathfrak{g}^\perp / Z(\mathfrak{g})$ and $Z(\mathfrak{g})$ are solvable, so $\mathfrak{g}^\perp$ is solvable by Lemma 4.8.

Thus $\mathfrak{g}^\perp \subset \mathrm{Rad}(\mathfrak{g}) = 0$, so $\mathfrak{g}^\perp = 0$, so κ is nondegenerate.

(iii) $\to$ (i):

Suppose the Killing form is nondegenerate.

Claim 1: $\mathfrak{g}$ has no nonzero abelian ideals.

If $I \subset \mathfrak{g}$ is such an ideal, and $x \in I$, $y \in \mathfrak{g}$, then

$$\mathrm{ad}(x) = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, \qquad , \mathrm{ad}(y) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \tag{4.7}$$

so $\mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) = 0$, so $x = 0$ because κ is nondegenerate. This proves the claim.

To show that $\mathfrak{g}$ is semisimple, we use induction on $\dim \mathfrak{g}$. If $\mathfrak{g}$ is simple, we are done. By Claim 1, it is not abelian. So let I be any proper, nonzero ideal and

$$J = \{x \in \mathfrak{g} \mid \kappa(x, y) = 0 \forall y \in I\} \tag{4.8}$$

be the orthogonal complement of I.

Claim 2: $\mathfrak{g} = I \oplus J$.

It suffices to show that $I \cap J = 0$. We have that $K = I \cap J$ is an ideal of $\mathfrak{g}$, and it is solvable by Cartan's theorem 3.27. Then $K^{(n)} = 0$ for some minimal n, so that $K^{(n-1)} \neq 0$. But then $K^{(n-1)}$ is abelian, which is a contradiction to claim 1. So $K = 0$.

Thus $\mathfrak{g} = I \oplus J$, and by inductive hypothesis, I and J are semisimple.

□

Lemma 4.12. *Every ideal and quotient of a semisimple Lie algebra is semisimple.*

Proof. Sheet 2. □

Note that this Lemma is not true for subalgebras.

4.4 Complete Reducibility

Let $\mathfrak{g}$ be a semisimple Lie algebra.

Theorem 4.13 (Weyl)**.** *Every representation of $\mathfrak{g}$ is completely reducible.*

Proof. The proof generalizes the $\mathfrak{sl}_2$ case. See Sheet 2. □

Lemma 4.14. *If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is faithful, then the trace form $(\cdot, \cdot)_V$ is nondegenerate.*

Proof. Cartan's trace criterion Theorem 3.27 shows that

$$\{x \in \mathfrak{g} \mid (x, y)_V = 0 \forall y \in \mathfrak{g}\} \tag{4.9}$$

is solvable.

Since $\mathrm{Rad}(\mathfrak{g}) = 0$, this shows that $(\cdot, \cdot)_V$ is nondegenerate. □

Definition 4.15. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a faithful representation. Pick a basis $x_1, \dots, x_m$ of $\mathfrak{g}$ and let $y_1, \dots, y_m$ be the dual basis with respect to the trace form, so that $(x_i, y_j)_V = \delta_{ij}$. Then the *Casimir element* of ρ is

$$\Omega_\rho = \sum_{i=1}^{m} \rho(x_i)\rho(y_i) \in \mathrm{End}(V) \tag{4.10}$$

Remark 4.16. The Casimir element is independent of the choice of basis of $\mathfrak{g}$, but we won't need this fact.

Example 4.17. If $\mathfrak{g} = \mathfrak{sl}_2$ and $\rho : \mathfrak{g} \to \mathfrak{gl}_2$ is the defining representation, then the Casimir element is defined as in (2.21):

$$\Omega_\rho = \rho(e)\rho(f) + \rho(f)\rho(e) + \frac{1}{2}\rho(h)^2 \in \mathfrak{gl}(V) = \mathrm{End}(V) \tag{4.11}$$

Proposition 4.18. *The Casimir element $\Omega_\rho : V \to V$ is $\mathfrak{g}$-equivariant, and $\mathrm{Tr}(\Omega_\rho) = \dim \mathfrak{g}$.*

Proof. Trace follows from direct calculation.

To show Ω_ρ is $\mathfrak{g}$-equivariant, let $x \in \mathfrak{g}$ and write

$$[x, x_i] = \sum_j a_{ij} x_j, \qquad [x, y_i] = \sum_j b_{ij} y_j. \tag{4.12}$$

Since $([x, x_i], y_j)_V = -(x_i, [x, y_j])_V$ by Lemma 4.6, we have that $a_{ij} = -b_{ji}$. Using that $[A, BC] = [A, B]C + B[A, C]$ for all $A, B, C \in \mathrm{End}(V)$, we have that $[\rho(x), \Omega_\rho] = 0$.

□

4.5 Jordan Decomposition

Let V be a vector space. Then an element $x \in \mathfrak{gl}(V) = \mathrm{End}(V)$ is semisimple if it is diagonalizable, so x has a basis of eigenvectors. x is *nilpotent* if $x^n = 0$ for some $n \geq 1$. Equivalently, x has all eigenvalues 0. So if x is both semisimple and nilpotent, then $x = 0$.

Proposition 4.19 (Concrete Jordan decomposition)**.** *Let $x \in \mathfrak{gl}(V)$. Then*

(i) There are unique $x_s, x_n \in \mathfrak{gl}(V)$ such that x_s is semisimple x_n is nilpotent, $x = x_s + x_n$, and $[x_s, x_n] = 0$. This is the Jordan decomposition *of x.*

(ii) Moreover, there exist polynomials $p_s, p_n \in \mathbb{C}[t]$ with zero constant term such that $x_s = p_s(x)$, and $x_n = p_n(x)$. These are not unique, and depend on x.

(iii) If $y \in \mathfrak{gl}(V)$ commutes with x, then it also commutes with x_s, x_n.

If $U, W \subset V$ are subspaces such that $x(U) \subset W$, then $x_s(U) \subset W$ and $x_n(U) \subset W$.

Proof. (ii) → (iii): If y commutes with x, it commutes with $p(x)$.

The other parts follows from routine Linear Algebra, see Humphreys §4.2 □

Example 4.20. If $V = \mathbb{C}^2$, $x = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ then $x = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ and $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

If $\lambda \neq 0$, then $p_s(t) = 2t - \lambda^{-1}t^2$ and $p_n(t) = \lambda^{-1}t^2 - t$ works.

The Jordan decomposition interacts nicely with semisimple subalgebras, as the next theorem shows.

Theorem 4.21. *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a semisimple subalgebra and $x \in \mathfrak{g}$ with Jordan decomposition $x = x_s + x_n$ in $\mathfrak{gl}(V)$. Then $x_s, x_n \in \mathfrak{g}$.*

Proof. This is just linear algebra, see Humphreys §6.4. □

Remark 4.22. This is not true for an arbitrary subalgebra of $\mathfrak{gl}(V)$. For instance, take the span of $x \in \mathfrak{gl}(V)$ such that $x \neq x_s, x_n$.

Now, if $\mathfrak{g}$ is a semisimple Lie algebra, then $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is injective (the adjoint representation is faithful) because the center is trivial. So by the above theorem, $\mathrm{ad}(x) = \mathrm{ad}(x_s) + \mathrm{ad}(x_n)$ is the Jordan decomposition of $\mathrm{ad}(x) \in \mathfrak{gl}(\mathfrak{g})$ for some *unique* (because of faithfulness) elements $x_s, x_n \in \mathfrak{g}$.

In other words, there exists a unique $x_s, x_n \in \mathfrak{g}$ such that $\mathrm{ad}(x) = \mathrm{ad}(x_s) + \mathrm{ad}(x_n)$ is the Jordan decomposition of $\mathrm{ad}(x)$. Since ad is a representation, we have that $x = x_s + x_n$, this is the *abstract Jordan decomposition* of x.

If $\mathfrak{g}$ is a semisimple subalgebra of $\mathfrak{gl}(V)$, there are two possible Jordan decompositions of $x \in \mathfrak{g}$: the concrete one from $\mathfrak{gl}(V)$, and the abstract one from the adjoint representation in $\mathfrak{gl}(\mathfrak{g})$.

Lemma 4.23. *If $x \in \mathfrak{gl}(V)$ has concrete Jordan decomposition $x = x_s + x_n$. Then $\mathrm{ad}(x) = \mathrm{ad}(x_s) + \mathrm{ad}(x_n)$ is the abstract Jordan decomposition of $\mathrm{ad}(x)$ in $\mathfrak{gl}(\mathfrak{gl}(V))$.*

Proof. We need to check that

(i) $\mathrm{ad}(x_s)$ is semisimple.

(ii) $\mathrm{ad}(x_n)$ is nilpotent.

(iii) $[\mathrm{ad}(x_s), \mathrm{ad}(x_n)] = 0$.

Since the abstract Jordan decomposition is unique, this shows that the concrete Jordan decomposition aligns with the abstract one.

(i) In a basis of eigenvectors for x_s, the elementary matrices in $\mathfrak{gl}(V)$ form a basis of eigenvectors for $\mathrm{ad}(x_s)$, so $\mathrm{ad}(x_s)$ is semisimple.

(ii) By Lemma 3.23, if $x_n \in \mathfrak{gl}(V)$ is nilpotent then $\mathrm{ad}(x_n) \in \mathfrak{gl}(\mathfrak{gl}(V))$ is nilpotent.

(iii) Since ad is a representation, we have that $[\mathrm{ad}(x_s), \mathrm{ad}(x_n)] = \mathrm{ad}([x_s, x_n]) = 0$. □

Corollary 4.24. *If $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a semisimple subalgebra and $x \in \mathfrak{g}$, then the concrete and abstract Jordan decompositions coincide.*

Proof. There's a proof of this in my notes but this appears to be immediate from the uniqueness of the abstract Jordan decomposition? □

From now on, we will simply refer to the *Jordan decomposition* of an element of a semisimple Lie algebra, since we have shown that the abstract and the concrete Jordan decompositions are the same. In the same way we can show the following.

Proposition 4.25. *If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of a semisimple Lie algebra $\mathfrak{g}$ and $x \in \mathfrak{g}$, then $\rho(x) = \rho(x_s) + \rho(x_n)$ is the Jordan decomposition of $\rho(x)$, where $x = x_s + x_n$ is the Jordan decomposition of x.*

Proof. Exercise. This should follows from the exact same argument as above. $\square$

If $\mathfrak{g}$ is semisimple, an element $x \in \mathfrak{g}$ is semisimple if $x = x_s$, so that $\mathrm{ad}(x)$ is a semisimple endomorphism, and it is nilpotent if $x = x_n$, so that $\mathrm{ad}(x)$ is nilpotent. So if $x \in \mathfrak{g}$ is diagonalizable, then $\rho(x)$ is diagonalizable for all representations ρ.

Example 4.26. $h \in \mathfrak{sl}_2(\mathbb{C})$ is diagonalizable, so $\rho(h)$ is diagonalizable for any representation ρ.

5 Root spaces: a new decomposition

In this section we introduce the notion of a *root* of a semisimple Lie algebra $\mathfrak{g}$. These are essential to the study of Lie algebras, and arise from the simultaneous diagonalization of a Cartan subalgebra, as described below.

5.1 The Cartan subalgebra

Let $\mathfrak{g}$ be a semisimple Lie algebra.

Definition 5.1. A subalgebra $\mathfrak{t} \subset \mathfrak{g}$ is *toral* if $\mathfrak{t}$ is abelian and for all $x \in \mathfrak{t}$, x is semisimple in $\mathfrak{g}$.

A toral subalgebra not contained in a bigger one is called a *maximal toral subalgebra*, or *Cartan subalgebra* (CSA).

We will sometimes refer to a toral subalgebra as a torus, and a CSA as a maximal torus.

Example 5.2. The set of diagonal matrices in $\mathfrak{sl}_n(\mathbb{C}), \mathfrak{sp}_{2\ell}(\mathbb{C}), \mathfrak{so}_n(\mathbb{C})$ are CSAs (Sheet 2).

Remark 5.3. We chose the particular matrix J when defining $\mathfrak{so}_n(\mathbb{C})$ and $\mathfrak{sp}_{2\ell}(\mathbb{C})$ so that the diagonal matrices are CSAs.

Remark 5.4. Later, we will see that CSAs are essentially unique, up to automorphism of $\mathfrak{g}$ (all maximal tori are conjugate).

Remark 5.5. The key to the classification of semisimple Lie algebras is to study the action of $\mathfrak{t}$ on $\mathfrak{g}$ via the adjoint representation, because $\mathfrak{t}$ is diagonalizable.

Lemma 5.6. *If $\mathfrak{h}$ is an abelian Lie algebra and $\rho : \mathfrak{h} \to \mathfrak{gl}(V)$ is a representation such that $\rho(x)$ is semisimple for all $x \in \mathfrak{h}$, then there exists $\lambda_1, \dots, \lambda_h \in \mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$ such that*

$$V = \bigoplus_{i=1}^{m} V_{\lambda_i} \tag{5.1}$$

where

$$V_{\lambda_i} = \{v \in V \mid xv = \lambda_i(x)v \forall x \in \mathfrak{h} \tag{5.2}$$

Proof. Choose a basis $t_1, \dots, t_m$ of $\mathfrak{h}$. Then $\rho(t_1), \dots, \rho(t_m)$ are commuting semisimple elements of $\mathfrak{gl}(V)$. Any $\lambda \in \mathfrak{h}^*$ is determined by $\lambda(t_1), \dots, \lambda(t_n) \in \mathbb{C}$. So the lemma follows from the fact that $\{\rho(t_1), \dots, \rho(t_n)\}$ can be simultaneously diagonalized because the elements are commuting and semisimple. See here. $\square$

5.2 The root space decomposition

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{t} \subset \mathfrak{g}$ a CSA, so a maximal abelian subalgebra of semisimple elements.

Consider the restriction of the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ to $\mathfrak{t}$. This is a representation of $\mathfrak{t}$. Since $\mathfrak{t}$ is made up of semisimple elements, the image of this representation consists of semisimple elements. By Lemma 5.6, we can simultaneously diagonalize, so there exists $S \subset \mathfrak{t}^*$ such that

$$\mathfrak{g} = \bigoplus_{\lambda \in S} \mathfrak{g}_\lambda, \tag{5.3}$$

where

$$\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid [t, x] = \lambda(t)x \ \forall t \in \mathfrak{t}\} \tag{5.4}$$

We partition S as $S = \{0\} \sqcup \Phi$, so that

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha. \tag{5.5}$$

This is called the *root space decomposition* of $\mathfrak{g}$, and the elements of Φ are called *roots*.

Example 5.7. Let $\mathfrak{g} = \mathfrak{sl}_2$ with basis $\{e, h, f\}$. Then $\mathfrak{t} = \mathrm{Span}\{h\}$ is a CSA (Sheet 2), so $\mathfrak{t}^* = \mathrm{Span}\{h^*\}$, where $h^* : h \mapsto 1$. Then $[h, e] = 2e$ and $[h, f] = -2f$, so $\Phi = \{\pm 2h^*\}$ and

$$\mathfrak{sl}_2 = \mathfrak{g}_0 \oplus \mathfrak{g}_{2h^*} \oplus \mathfrak{g}_{-2h^*} \tag{5.6}$$

where $\mathfrak{g}_0$ is the span of h, $\mathfrak{g}_{2h^*}$ is the span of e, and $\mathfrak{g}_{-2h^*}$ is the span of f.

Example 5.8. Let $\mathfrak{g} = \mathfrak{sl}_3$. Then the space diagonal matrices with zero trace is a CSA, which we denote by $\mathfrak{t}$. Letting $e_i^* \in \mathfrak{t}^*$ be given by

$$e_i \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} = t_i, \tag{5.7}$$

then $\mathfrak{t}^*$ is spanned by $\{e_i^*\}$ with relations $e_1^* + e_2^* + e_3^* = 0$. We can compute that

$$\left[\begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix}, E_{ij} \right] = (t_i - t_j)E_{ij} \tag{5.8}$$

so that

$$\Phi = \{e_i^* - e_j^* \mid 1 \le i \ne j \le 3\} \tag{5.9}$$

so

$$\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_{e_1-e_2} + \mathfrak{g}_{e_1-e_3} + \mathfrak{g}_{e_2-e_3} + \mathfrak{g}_{e_2-e_1} + \mathfrak{g}_{e_3-e_1} + \mathfrak{g}_{e_3-e_2} \tag{5.10}$$

with the root space $\mathfrak{g}_{e_i-e_j}$ spanned by the matrix E_{ij}. Note that we are somewhat "lucky" that our simultaneous diagonalization of $\mathfrak{sl}_3$ gives a basis of elementary matrices.

It is not a coincidence that all the nonzero root spaces are one-dimensional, however, as this is always the case, as we shall see.

5.3 First properties of roots

Lemma 5.9. *If $\alpha, \beta \in \mathfrak{t}^*$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.*

Proof. Direct calculation. Use Jacobi identity. $\square$

The next lemma is very important! Using the next lemma and the its nondegeneracy, we are able to get a handle on the Killing form, which has been a bit mysterious up to this point.

Lemma 5.10. *If $\alpha, \beta \in \mathfrak{t}^*$ are such that $\alpha + \beta \neq 0$, then $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$ under the Killing form.*

Proof. There exists $t \in \mathfrak{t}$ such that $(\alpha + \beta)(t) \neq 0$. If $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, then

$$\begin{aligned} \kappa([x,t],y) &= \alpha(t)\kappa(x,y) \\ &= -\kappa(x,[t,y]) \\ &= -\beta(t)\kappa(x,y) \end{aligned} \tag{5.11}$$

so $(\alpha+\beta)(t)\kappa(x,y) = 0$ so $\kappa(x,y) = 0$. $\square$

Corollary 5.11. *The restriction of κ to $\mathfrak{g}_0$ is nondegenerate.*

Proof. We have that $\mathfrak{g}_0 \perp \mathfrak{g}_\alpha$ for all $\alpha \in \Phi$ since $\alpha \neq 0$, so if $x \in \mathfrak{g}_0$ satisfies $\kappa(x,y) = 0$ for all $y \in \mathfrak{g}_0$, then $\kappa(x,y) = 0$ for all $y \in \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha = \mathfrak{g}$, so then $x = 0$ because κ is nondegenerate. $\square$

5.4 The zero weight space

We have that the zero weight space

$$\mathfrak{g}_0 = \{x \in \mathfrak{g} \mid [t,x] = 0 \forall t \in \mathfrak{t}\} \tag{5.12}$$

is the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$. Since $\mathfrak{t}$ is abelian, we have that $\mathfrak{t} \subset \mathfrak{g}_0$.

Proposition 5.12. *We have that $\mathfrak{t} = \mathfrak{g}_0$.*

Proof. Omitted, see Humphreys §8.2. $\square$

Combining this with Corollary 5.11, we find that $\kappa|_{\mathfrak{t}\times\mathfrak{t}} : \mathfrak{t} \times \mathfrak{t} \to \mathbb{C}$ is nondegenerate. Therefore the map $\mathfrak{t} \to \mathfrak{t}^*$ given by $x \mapsto \kappa(x, \cdot)$ is an isomorphism.

Definition 5.13. If $\lambda \in \mathfrak{t}^*$, let t_λ be the unique element of $\mathfrak{t}$ such that $\kappa(t_\lambda, x) = \lambda(x)$ for all $x \in \mathfrak{t}$.

5.5 Finding $\mathfrak{sl}_2$s in $\mathfrak{g}$

We will show that for every $\alpha \in \Phi$, we can find a subalgebra $\mathfrak{m}_\alpha \subset \mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$ such that $\mathfrak{g}_\alpha \subset \mathfrak{m}_\alpha$.

Lemma 5.14. *We have that*

(i) *Φ spans $\mathfrak{t}^*$.*

(ii) *If $\alpha \in \Phi$, then $-\alpha \in \Phi$.*

Proof. (i) If Φ spans $\mathfrak{t}^*$, then there is no nonzero $t \in \mathfrak{t}$ vanishing at all $\alpha \in \Phi$. Let $t \in \mathfrak{t}$ be an element such that $\alpha(t) = 0$ for all $\alpha \in \Phi$. It suffices to show that $t = 0$. We know that $[t, x] = \alpha(t)x = 0$ for all $x \in \mathfrak{g}_\alpha, \alpha \in \Phi$. Since $[t, x] = 0$ for all $x \in \mathfrak{g}$, we have that $t \in Z(\mathfrak{g}) = 0$ since $\mathfrak{g}$ is semisimple.

(ii) Since $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$ if $\alpha + \beta \neq 0$, we have that $\mathfrak{g}_\alpha$ is perpendicular to all the root spaces with $\alpha + \lambda \neq 0$. But since the Killing form is nondegenerate, we must have that $-\alpha \in \Phi$. □

Proposition 5.15. *We have that*

(i) If $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$, then $[x, y] = t_\alpha \kappa(x, y)$.

(ii) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one-dimensional.

(iii) $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$.

Proof. (i)The identity is equivalent to $\kappa([x, y], t) = \alpha(t)\kappa(x, y)$ for all $t \in \mathfrak{t}$, as then $[x, y]/\kappa(x, y)$ is the unique $t_\alpha \in \mathfrak{t}$ such that $\kappa(t_\alpha, t) = \alpha(t)$.

We have that

$$\kappa([x, y], t) = \kappa(x, [y, t]) = \kappa(x, \alpha(t)y) = \alpha(t)\kappa(x, y) \tag{5.13}$$

as desired.

(ii) By (i), it suffices to show that $\kappa(x, y) \neq 0$ for some $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$, as then we have that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is spanned by $t_\alpha \kappa(x, y) \neq 0$.

For every $x \in \mathfrak{g}_\alpha \setminus \{0\}$, there exists $y \in \mathfrak{g}_{-\alpha}$ such that $\kappa(x, y) \neq 0$ as otherwise κ would be degenerate.

(iii) Let $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\alpha$ be elements such that $\kappa(x, y) \neq 0$ (these exist by (ii)). After scaling, we may assume that $\kappa(x, y) = 1$, so then $[x, y] = t_\alpha$, and $[t_\alpha, x] = \alpha(t_\alpha)x$ and $[t_\alpha, y] = -\alpha(t_\alpha)y$. So

$$\mathfrak{h} = \operatorname{Span}\{x, t_\alpha, y\} \tag{5.14}$$

is a subalgebra. Suppose for the sake of contradiction that $\alpha(t_\alpha) = 0$. Then $\mathfrak{h}$ is solvable as $\mathfrak{g}^{(2)} = 0$.

By Lie's theorem, there is basis of $\mathfrak{g}$ such that $\operatorname{ad} : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$ lands in the space of upper triangular matrices. So then $t_\alpha \in [\mathfrak{h}, \mathfrak{h}]$ has the property that $\operatorname{ad}(t_\alpha) \in \mathfrak{gl}(\mathfrak{g})$ is strictly upper triangular, so it is nilpotent. Since it is also semisimple, we have that $\operatorname{ad}(t_\alpha) = 0$, so $t_\alpha = 0$, which is a contradiction. □

We are ready to construct our $\mathfrak{sl}_2$ triple $\mathfrak{m}_\alpha$. For every $\alpha \in \Phi$, define

$$h_\alpha := \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)} \in \mathfrak{t}. \tag{5.15}$$

This will serve the roll of $h \in \mathfrak{sl}_2$ in our $\mathfrak{sl}_2$ triple.

Proposition 5.16. *If $\alpha \in \Phi$ and $e_\alpha \in \mathfrak{g}_\alpha$ is any nonzero element, then there exists $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $(e_\alpha, h_\alpha, f_\alpha)$ is a triple satisfying the $\mathfrak{sl}_2$ relations:*

$$[h_\alpha, e_\alpha] = 2e_\alpha, \quad [h_\alpha, f_\alpha] = -2f_\alpha, \quad [e_\alpha, f_\alpha] = h_\alpha. \tag{5.16}$$

Proof. Since the Killing form is nondegenerate and $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$ for $\alpha + \beta \neq 0$, there exists $f_\alpha \in g_{-\alpha}$ such that $\kappa(e_\alpha, f_\alpha) = 2/\kappa(t_\alpha, t_\alpha)$ after rescaling. By Proposition 5.15 (i), we have that $[e_\alpha, f_\alpha] = \kappa(e_\alpha, f_\alpha)t_\alpha = h_\alpha$. We also have that

$$[h_\alpha, e_\alpha] = \alpha(h_\alpha)e_\alpha = 2e_\alpha \tag{5.17}$$

since

$$\alpha(h_\alpha) = \alpha\left(\frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}\right) = \frac{2\alpha(t_\alpha)}{\kappa(t_\alpha,t_\alpha)} = 2 \tag{5.18}$$

since $\alpha(t_\alpha) = \kappa(t_\alpha,t_\alpha)$ by Proposition 5.15 (iii). Similarly, $[h_\alpha, f_\alpha] = -2f_\alpha$. $\square$

Thus if $(e_\alpha, h_\alpha, f_\alpha)$ satisfies the conclusions of the Proposition, letting $m_\alpha = \mathrm{Span}(e_\alpha, h_\alpha, f_\alpha)$, we have that m_α is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$. We call $(e_\alpha, h_\alpha, f_\alpha)$ a $\mathfrak{sl}_2$ *triple*. Thus $\mathfrak{g}$ is made up of a bunch of $\mathfrak{sl}_2$s glued together in a nice way.

5.6 Root strings

Proposition 5.17. *If $\alpha \in \Phi$, then $\dim \mathfrak{g}_\alpha = 1$. Moreover, if $c \in \mathbb{C}$, then $c\alpha \in \Phi$ if and only if $c = \pm 1$.*

Proof. Choose an $\mathfrak{sl}_2$ triple $(e_\alpha, h_\alpha, f_\alpha)$ giving rise to $\mathfrak{m}_\alpha \subset \mathfrak{g}$. Let

$$V = \mathfrak{t} \oplus \bigoplus_{c\in\mathbb{C}} \mathfrak{g}_{c\alpha} \tag{5.19}$$

which is a subspace of $\mathfrak{g}$. In fact, V is stable under the adjoint action of $\mathfrak{m}_\alpha$, so that $[\mathfrak{m}_\alpha, V] \subset V$. This follows from $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. So V is a representation of $\mathfrak{m}_\alpha \cong \mathfrak{sl}_2$ under the adjoint action. The weights of V (recall how we defined the weights of a $\mathfrak{sl}_2$ representation in Section 2.4 are the eigenvalues of h_α. But since $h_\alpha \in \mathfrak{t}$, these are easy to describe. On $\mathfrak{t}$, the weights are 0 with multiplicity $\dim \mathfrak{t}$.

On $\mathfrak{g}_{c\alpha}$, the weights are $(c\alpha)(h_\alpha) = c\alpha(h_\alpha) = 2c$ with multiplicity $\dim \mathfrak{g}_{c\alpha}$. Thus $2c \in \mathbb{Z}$ if $c\alpha \in \Phi$, as all the weights are integers. Moreover, $U = \mathfrak{t} + \mathfrak{m}_\alpha \subset V$ is an m_α-subrepresentation, because $[\mathfrak{t}, \mathfrak{m}_\alpha] \subset \mathfrak{m}_\alpha$, and $\mathfrak{m}_\alpha$ is a subalgebra.

By complete reducibility, there exists $W \subset V$ such that $V = U \oplus W$. What are the weights of W? 0 is *not* a weight of W, since the 0 weight space of V is contained in U. So by the representation theory of $\mathfrak{sl}_2$, W has no even weights. So $2\alpha \notin \Phi$, because then 4 would be a weight of W (2 is a V associated with $\mathfrak{g}_\alpha$). So $2\alpha \notin \Phi$ for all $\alpha \in \Phi$. Then $\alpha/2 \notin \Phi$ for all $\alpha \in \Phi$, because then we would have $2(\alpha/2) = \alpha \notin \Phi$. Since $\mathfrak{g}_{\alpha/2}$ has weight 1 W has no weight 1, so it has no odd weights. Thus $W = 0$. So then $V = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = \mathfrak{t} \oplus \mathrm{Span}(e_\alpha, f_\alpha)$. $\square$

Corollary 5.18. *If $\alpha \in \Phi$, then $\mathfrak{m}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, -\mathfrak{g}_\alpha]$.*

Proof. We have that $\mathfrak{g}_\alpha = \mathrm{Span}\, e_\alpha$, $\mathfrak{g}_{-\alpha} = \mathrm{Span}\, f_\alpha$, and $[g_\alpha, g_{-\alpha}] = \mathrm{Span}\, h_\alpha$. $\square$

Let $\alpha, \beta \in \Phi$, and assume $\beta \neq \pm\alpha$. Let

$$V = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{\beta+k\alpha} \tag{5.20}$$

This is preserved under the adjoint action of $\mathfrak{m}_\alpha$ since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

Thus V is a representation of $m_\alpha \cong \mathfrak{sl}_2$. What are the weights of V? If $x \in \mathfrak{g}_{\beta+k\alpha}$, then

$$[h_\alpha, x] = (\beta + k\alpha)(h_\alpha)x \tag{5.21}$$

On $\mathfrak{g}_{\beta+k\alpha}$, h_α has weight

$$(\beta + k\alpha)(h_\alpha) = \beta(h_\alpha) + k\alpha h(\alpha) = \beta(h_\alpha) + 2k \tag{5.22}$$

with multiplicity 1, since $\mathfrak{g}_{\beta+k\alpha}$ has dimension 1 as $\beta + k\alpha \neq 0$.

So V has roots of the same parity and multiplicity 1, so V is irreducible. Then $V \cong V(n)$ for some $n \in \mathbb{Z}_{\geq 0}$, so it has weights $\{n, n-2, \ldots, -n\}$. The set $\{\beta + k\alpha \mid k \in \mathbb{Z}\} \cap \Phi$ is called the *α-root string* of β.

Proposition 5.19. *(i) If α, β are roots with $\alpha \neq \pm\beta$. Let $r \geq 0$ be the largest integers such that $\beta - r\alpha \in \Phi$ and let $q \geq 0$ be the largest integer such that $\beta + q\alpha \in \Phi$. Then*

$$\{\beta + k\alpha \mid k \in \mathbb{Z}\} \cap \Phi = \{\beta + k\alpha \mid -r \leq k \leq q\} \tag{5.23}$$

and $\beta(h_\alpha) = r - q \in \mathbb{Z}$.

(ii) $\beta - \beta(h_\alpha)\alpha \in \Phi$.

(iii) If $\alpha + \beta \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

Proof. We use that $V \cong V(n)$ where V is defined as in (5.20). Using this fact and $\mathfrak{sl}_2$-theory gives the result.

(i) We have that $n = (\beta + q\alpha)(h_\alpha)$ and $-n = (\beta - r\alpha)(h_\alpha)$ so $\beta(h_\alpha) = r - q$.

Also, since the weights of V include n, $-n$, they must include the weights "in between".

(ii) Note that $-r \leq -\beta(h_\alpha) \leq q$, so we can conclude this by part (i).

(iii) This follows from the fact that if $\lambda, \lambda + 2$ are weights of an $\mathfrak{sl}_2$-representation, then $e : V_\lambda \to V_{\lambda+2}$ is surjective. So $\mathfrak{g}_\alpha$ acting on $\mathfrak{g}_\beta$ gives $\mathfrak{g}_{\alpha+\beta}$. ◻

5.7 Φ is a root system

The above results show that if $\alpha, \beta \in \Phi$, then $\beta(h_\alpha) \in \mathbb{Z}$, and $\beta - \beta(h_\alpha)\alpha \in \Phi$, and if $\beta = \pm\alpha$, then $\beta(h_\alpha) = \pm 2$. These conditions are enough to show that Φ is a *root system*, which is essentially a very nice collection of real vectors.

Recall that $\kappa : \mathfrak{t} \times \mathfrak{t} \to \mathbb{C}$ is nondegenerate. So $\mathfrak{t} \to \mathfrak{t}^*$, $x \to \kappa(x, \cdot)$ is an isomorphism with inverse $\lambda \mapsto t_\lambda$. We can use this to define a dual pairing on $\mathfrak{t}^*$ by

$$\begin{aligned} (\cdot, \cdot) : \mathfrak{t}^* \times \mathfrak{t}^* &\to \mathbb{C} \\ (\lambda, \mu) &\mapsto \kappa(t_\lambda, t_\mu) \end{aligned} \tag{5.24}$$

Because κ is nondegenerate, this is a nondegenerate pairing.

Now, since Φ spans $\mathfrak{t}^*$, there is a basis of roots $\alpha_1, \ldots, \alpha_\ell$.

Lemma 5.20. *If $\beta \in \Phi$, then $\beta = \sum_{i=1}^{\ell} c_i \alpha_i$ with $c_i \in \mathbb{Q}$ for all i.*

Proof. We have that $\beta = \sum_{i=1}^{\ell} c_i \alpha_i$ with $c_i \in \mathbb{C}$. For each j, calculating (α_j, β) and scaling gives

$$\frac{2(\alpha_j, \beta)}{(\alpha_j, \alpha_j)} = \sum_{i=1}^{\ell} c_i \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \tag{5.25}$$

This is a system of linear equations in $c_1, \ldots, c_\ell$. Since the form is nondegenerate, $((\alpha_i, \alpha_j))_{ij}$ is invertible, so $\left(\frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}\right)_{ij}$ is also invertible.

Moreover, if $\alpha, \beta \in \Phi$, then

$$\frac{2(\beta,\alpha)}{(\alpha,\alpha)} = \frac{2\kappa(t_\beta, t_\alpha)}{\kappa(t_\alpha, t_\alpha)} = \kappa(t_\beta, h_\alpha) = \beta(h_\alpha) \in \mathbb{Z} \tag{5.26}$$

so the coefficients of the matrix are in $\mathbb{Q}$, so the solution (c_i) is also in $\mathbb{Q}$. $\square$

Now, let $E = \mathbb{R}\Phi \subset \mathfrak{t}^*$ be the $\mathbb{R}$-span of Φ in $\mathfrak{t}^*$. By the lemma, $\dim_{\mathbb{R}} E = \dim \mathbb{C}\mathfrak{t}^*$ since E has an $\mathbb{R}$-basis $\{\alpha_1, \dots, \alpha_\ell\}$. The next theorem shows that the collection $\Phi \subset E$ satisfies the axioms of a *root system.*

Theorem 5.21. *(i) The restriction of $(\cdot,\cdot)$ to $E \times E$ is real valued and positive definite.*

(ii) If $\alpha \in \Phi$ and $c \in \mathbb{R}$, then $c\alpha \in \Phi$ if and only if $c = \pm 1$.

(iii) If $\alpha, \beta \in \Phi$, then $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi$.

(iv) If $\alpha, \beta \in \Phi$, then $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Proof. (ii), (iii), (iv) are done by Propositions 5.17 and 5.19 since $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} = \beta(h_\alpha)$.

(i): We show that $(\alpha,\beta) \in \mathbb{Q}$ for all $\alpha, \beta \in \Phi$. If $\lambda, \mu \in \mathfrak{t}^*$, then $(\lambda,\mu) = \kappa(t_\lambda, t_\mu) = \mathrm{Tr}(\mathrm{ad}(t_\lambda)\,\mathrm{ad}(t_\mu))$ which is the sum of the eigenvalues of $\mathrm{ad}(t_\lambda)\,\mathrm{ad}(t_\mu)$. Now, $\mathrm{ad}(t_\lambda)$ has eigenvalue 0 on $\mathfrak{t}$, and $\alpha(t_\lambda)$ on $\mathfrak{g}_\alpha$. So

$$(\lambda,\mu) = \sum_{\alpha\in\Phi} \alpha(t_\lambda)\alpha(t_\mu) = \sum_{\alpha\in\Phi} (\lambda,\alpha)(\mu,\alpha) \tag{5.27}$$

since $(\lambda,\alpha) = \kappa(t_\lambda, t_\alpha) = \alpha(t_\lambda)$. We plug in $\lambda = \mu = \beta \in \Phi \setminus \{0\}$ to get $(\beta,\beta) = \sum_{\alpha\in\Phi}(\alpha,\beta)^2$ so

$$\frac{1}{(\beta,\beta)} = \frac{1}{(\beta,\beta)^2} \sum_{\alpha\in\Phi} \frac{1}{4}\alpha(h_\beta)^2(\beta,\beta)^2 = \sum_{\alpha\in\Phi} \frac{1}{4}\alpha(h_\beta)^2 \in \mathbb{Q}_{>0} \tag{5.28}$$

by (iv) and the non degeneracy of the form, which gives positive definiteness

Also, $(\alpha,\beta) = (\beta,\beta)\cdot\frac{(\alpha,\beta)}{(\beta,\beta)} = (\beta,\beta) = (\beta,\beta)\frac{1}{2}\alpha(h_\beta) \in \mathbb{Q}$. $\square$

6 Root spaces: the abstract strikes back

We have shown that Φ is a root system. Root systems can be classified, and the root system of a semisimple Lie algebra determines it. Thus to classify semisimple Lie algebras, we can classify root systems in the abstract, which we now do.

Definition 6.1. A *Euclidean space* $(E, (\cdot,\cdot))$ is a real vector space E and a positive definite bilinear form (an inner product).

If $\lambda, \alpha \in E$ and $\alpha \neq 0$, define

$$\langle \lambda, \alpha^\vee \rangle = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \tag{6.1}$$

and the reflection map

$$\begin{aligned} w_\alpha : E &\to E \\ \lambda &\mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \end{aligned} \tag{6.2}$$

This is the reflection along the hyperplane

$$H_\alpha \coloneqq \{x \in E \mid (x, \alpha) = 0\} \tag{6.3}$$

It sends $\alpha \mapsto -\alpha$, and is the identity on H_α, so it is a reflection. We have that $w_\alpha^2 = 1$.

Definition 6.2. A *root system* is a finite subset Φ of a Euclidean space E such that

(R1) Φ spans E and $0 \notin \Phi$.

(R2) If $\alpha \in \Phi$ and $c \in \mathbb{R}$, then $c\alpha \in \Phi$ if and only if $c\alpha = \pm 1$.

(R3) For all $\alpha \in \Phi$, $w_\alpha(\Phi) = \Phi$ (reflection preserves roots).

(R4) For all $\alpha, \beta \in \Phi$, $\langle \beta, \alpha^\vee \rangle = \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$. So the reflection of β onto the line spanned by α is an integer of half-integer multiple of α.

The *rank* of Φ is $\dim_{\mathbb{R}} E$.

An isomorphism of root systems Φ, Φ' is an isomorphism of vector spaces $f : E \to E'$ such that $f(\Phi) = \Phi'$ and $\langle f(\beta), f(\alpha)^\vee \rangle = \langle \beta, \alpha^\vee \rangle$ for all $\alpha, \beta \in \Phi$.

6.1 Examples of root systems

If $\mathfrak{t}$ is a CSA of a semisimple root system $\mathfrak{g}$, then $\Phi = \Phi(\mathfrak{g}, \mathfrak{t}) \subset E = \mathbb{R}\Phi$ is a root system, as we have shown in the previous section.

Example 6.3. Let $E = \mathbb{R}$ with the standard inner product, and $\Phi = \{\pm 1\} \subset E$. If $\alpha = \pm 1 \in E$, then $\langle \alpha, \alpha^\vee \rangle = 2$ and $\langle -\alpha, \alpha^\vee \rangle = -2$. This is the A_1 root system.

Example 6.4. The root systems of rank 2 are given in Figure 1. It is easy to verify that they satisfy the root system axioms.

Definition 6.5. The *Weyl group*, denoted by $W(\Phi)$ or just W, is the subgroup of $\mathrm{GL}(E)$ generated by $\{w_\alpha \mid \alpha \in \Phi\}$.

Lemma 6.6. *$W(\Phi)$ is finite.*

Proof. Since Φ spans E and w_α preserves Φ, $W(\Phi)$ acts faithfully on Φ, so it permutes the finite number of roots, so it is finite. $\square$

Example 6.7.

1. If $\Phi = A_1$, then $W(\Phi) = \{1, w_\alpha\} \cong C_2$, because $w_\alpha = w_{-\alpha}$.
2. We have that $W(A_1 \times A_1) \cong C_2 \times C_2$.
3. $W(A_2)$ contains $r = W_\alpha W_\beta$. r sends $\alpha \to \beta$ and $\beta \to -(\alpha + \beta)$ so r is rotation by $120°$, so $r^3 = 1$. So $W(A_2)$ is generated by $s = w_\alpha$ and $r = w_\alpha w_\beta$ so $W(A_2) \cong D_3$ is the dihedral group.

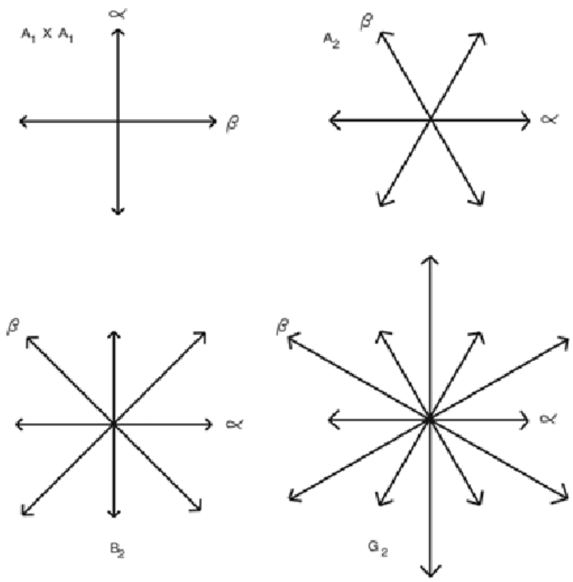

Figure 1: The root systems of rank 2.

4. $W(B_2) \cong D_4$, $W(G_2) \cong D_6$.

5. Let $n \geq 1$ and
$$E = \{(x_i) \subseteq \mathbb{R}^{n+1} \mid \sum_{x_i} = 0\} \tag{6.4}$$
with the standard inner product. Let
$$\Phi = \{e_i - e_j \mid 1 \leq i \neq j \leq n+1\} \tag{6.5}$$
where e_i is the standard basis for $\mathbb{R}^{n+1}$. This is a root system, called the root system of type A_n. If $\alpha = e_i - e_j$, then w_α swaps the i and j coordinates. Moreover, $(\alpha, \alpha) = 2$ for all $\alpha \in \Phi$, so $\langle \beta, \alpha^\vee \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} = (\beta, \alpha) \in \mathbb{Z}$. Since transpositions generate S_{n+1}, we have that $W(A_n) \cong S_{n+1}$.

6.2 Irreducible root systems

Definition 6.8. A root system (Φ, E) is *reducible* if there is a partition $\Phi = \Phi_1 \sqcup \Phi_2$ such that $\Phi_1, \Phi_2 \neq \emptyset$ and $\Phi_1 \perp \Phi_2$.

If Φ is not reducible, it is *irreducible*.

Example 6.9. $A_1 \times A_1$ is reducible, A_2, B_2, G_2 are irreducible.

If Φ is reducible, so that $\Phi = \Phi_1 \sqcup \Phi_2$ with $\Phi_1 \perp \Phi_2$, let $E_i = \mathrm{Span}_{\mathbb{R}} \Phi_i$. Then $E = E_1 \oplus E_2$ with $E_1 \perp E_2$ and Φ_i is a root system in E_i.

6.3 Angles

Let (Φ, E) be a root system. The integrality condition R4 turns out to be very restrictive.

Lemma 6.10. *If $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$, then $\langle \beta, \alpha^\vee \rangle \cdot \langle \alpha, \beta^\vee \rangle \in \{0, 1, 2, 3\} l$*

Proof. If $v \in E$ then $||v|| = (v, v)^{1/2}$. If θ is the angle between α, β, then $(\alpha, \beta) = (\cos\theta)||\alpha||||\beta||$. So

$$\langle \beta, \alpha^\vee \rangle \cdot \langle \alpha, \beta^\vee \rangle = \frac{4\langle \alpha, \beta \rangle^2}{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle} = 4\cos^2\theta \in \mathbb{Z} \tag{6.6}$$

But $0 \leq |\cos\theta| < 1$, so $4\cos^2\theta = \{0, 1, 2, 3\}$. ☐

Now, let $\alpha, \beta \in \Phi$ be roots such that $(\beta, \beta) \geq (\alpha, \alpha)$ and $\beta \neq \pm\alpha$. Then we can list all the options for $(\beta, \alpha^\vee)$:

$(\beta, \alpha^\vee)$	$(\alpha, \beta^\vee)$	$\frac{(\beta,\beta)}{(\alpha,\alpha)}$	θ
0	0	?	$\pi/2$
1	1	1	$\pi/3$
-1	-1	1	$2\pi/3$
2	1	2	$\pi/4$
-2	-1	2	$3\pi/4$
3	1	3	$\pi/6$
-3	-1	3	$5\pi/6$

Using this, we can classify all rank 2 root systems (exercise), and see that they are the ones given in Figure 1.

Corollary 6.11. *If $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$, and $(\alpha, \beta) < 0$, then $\alpha + \beta \in \Phi$.*

Proof. WLOG assume $(\beta, \beta) \geq (\alpha, \alpha)$. Then $(\alpha, \beta^\vee) = -1$ by the table above so $w_\beta(\alpha) = \alpha + \beta \in \Phi$. ☐

Corollary 6.12. *If Φ is irreducible, then $\{(\alpha, \alpha) \mid \alpha \in \Phi\}$ has size at most 2.*

Proof. Exercise. ☐

Corollary 6.13. *Root strings have size at most 4.*

Proof. The root string has length $(\beta, \alpha^\vee) + 1$, which we know is at most 4 by the table. ☐

6.4 Root bases

We want to express roots as positive integer linear combinations of a set of "basis roots".

Definition 6.14. $\Delta \subset \Phi$ is a *root basis* if

(i) Δ is an $\mathbb{R}$-basis of E.

(ii) Writing $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$, then for every $\alpha \in \Phi$ we can write

$$\alpha = \sum_{i=1}^{\ell} c_i \alpha_i \tag{6.7}$$

where $c_i \in \mathbb{Z}$ and either $c_i \geq 0$ for all i for $c_i \leq 0$ for all i.

We call the elements of Δ *simple roots* and the elements $\alpha \in \Phi$ with $c_i \geq 0$ are called *positive roots*, denoted by $\Phi^+ \subset \Phi$.

Example 6.15. In A_2, $\{\alpha, \beta\}$ is a root basis, and $\Phi^+ = \{\alpha, \beta, \alpha+\beta\}$.

More generally, in A_n, we have a root basis $\Delta = \{e_1 - e_2, \ldots, e_n - e_{n+1}\}$ with positive roots $\{e_i - e_j \mid i < j\}$.

If $\alpha \in \Phi$, let $H_\alpha = \{x \in E \mid (x, \alpha) = 0\}$ be the hyperplane perpendicular to α. By dimensionality, $E \setminus \bigcup_{\alpha \in \Phi} H_\alpha \neq \emptyset$. Take $\gamma \in E \setminus \bigcup_{\alpha \in \Phi} H_\alpha$ and define

$$\Phi_\gamma^+ := \{\alpha \in \Phi \mid (\alpha, \gamma) > 0\}, \qquad \Phi_\gamma^- := \{\alpha \in \Phi \mid (\alpha, \gamma) < 0\}. \tag{6.8}$$

Then $\Phi = \Phi_\gamma^+ \sqcup \Phi_\gamma^-$. We say that an element $\alpha \in \Phi_\gamma^+$ is *decomposable* if $\alpha = \beta_1 + \beta_2$ with $\beta_1, \beta_2 \in \Phi_\gamma^+$ and it is *indecomposable* otherwise.

Let Δ_γ be the set of indecomposable elements of Φ_γ^+.

Theorem 6.16. *Δ_γ is a root basis with positive roots Φ_γ^+. Moreover, every root basis is of the form Δ_γ for some $\gamma \in E \setminus \bigcup_{\alpha \in \Phi} H_\alpha$.*

Proof. We prove 3 claims:

Claim 1. If $\Delta_\gamma = \{\alpha_1, \ldots, \alpha_\ell\}$ then every element of Φ_γ^+ is of the form $\sum c_i \alpha_i$ with $c_i \in \mathbb{Z}_{\geq 0}$.

We show this by contradiction. Let $\alpha \in \Phi_\gamma^+$ be a counter example with (α, γ) minimal. Then α is decomposable, so $\alpha = \beta_1 + \beta_2$. But $(\alpha, \gamma) = (\beta_1, \gamma) + (\beta_2, \gamma)$ so $(\beta_1, \gamma) < (\alpha, \gamma)$ and $(\beta_2, \gamma) < (\alpha, \gamma)$, so $\beta_1 = \sum c_i \alpha_i$ and $\beta_2 = \sum d_i \alpha_i$ with $c_i, d_i \geq 0$ so $\alpha = \sum (c_i + d_i)\alpha_i$, which is a contradiction as $c_i + d_i \geq 0$

Claim 2. If $\alpha, \beta \in \Delta_\gamma$ and $\alpha \neq \beta$, then $(\alpha, \beta) \leq 0$.

Indeed, recall that if $(\alpha, \beta) \in \Phi$ with $\alpha \neq \pm\beta$ and $(\alpha, \beta) < 0$, then $\alpha + \beta \in \Phi$. Therefore if $(\alpha, \beta) > 0$, then $\alpha - \beta \in \Phi$, so $\alpha = \beta + (\alpha - \beta)$ is decomposable after possibly switching α and β so that $(\alpha - \beta, \gamma) > 0$. Thus $(\alpha, \beta) \leq 0$.

Claim 3. Elements of Δ_γ are linearly independent.

We'll show that if $S = \{\lambda_1, \ldots, \lambda_n\} \subset E$ has $(\lambda_i, \gamma) > 0$, and $(\lambda_i, \lambda_j) \leq 0$ for all $i \neq j$, then S consists of linearly independent elements. If $\sum c_i \lambda_i = 0$, we can write

$$e = \sum_{i \leq m} c_i \lambda_i = \sum_{i > m} c_i \lambda_i \tag{6.9}$$

with $c_i \geq 0$ (split the negative and postive c_is and relabel). Then

$$(e, e) = \sum_{i \leq m} \sum_{j > m} c_i c_j (\lambda_i, \lambda_j) \leq 0 \tag{6.10}$$

so $e = 0$.

Then $(e, \gamma) = \sum_{i \leq m} c_i (\lambda_i, \gamma) = \sum_{i > m} c_i (\lambda_i, \gamma) \geq 0$ so $c_i = 0$ for all i so S is a linearly independent set.

Proof of main claim. Now we are ready to prove the theorem and show that Δ_γ is a root basis.

Since Φ spans E, Φ_γ^+ spans E, and by Claim 1, Δ_γ spans E. By Claim 3, Δ_γ is a basis.

By Claim 1, and the fact that $\Phi = \Phi_\gamma^+ \sqcup \Phi_\gamma^-$, Δ_γ is a root basis.

Now, if $\Delta \subset \Phi$ is any root basis, we want to choose a vector $\gamma \in E$ such that $(\gamma, \alpha) > 0$ for all $\alpha \in \Delta$. Let $v_1, \dots, v_n$ be the dual basis with respect to $(\cdot, \cdot)$ and take $\gamma = v_1 + \cdots v_n$. Then for $\alpha = \sum c_i \alpha_i \in \Phi^+$, we have that $(\gamma, \alpha) = \sum c_i > 0$, so $\Phi^+ \subseteq \Phi_\gamma^+$. Since both sets have the same cardinality, we have that $\Phi^+ = \Phi_\gamma^+$.

So we just need to show that every element of Δ is indecomposable as an element of Φ_γ^+. If $\alpha \in \Delta$ is of the form $\alpha = \beta_1 + \beta_2$ with $\beta_1, \beta_2 \in \Phi^+$, then $\beta_i = \sum c_{ij} \alpha_j$ with $c_{ij} \geq 0$. So $\alpha = \sum_j (c_{1j} + c_{2j}) \alpha_j$. But since $\alpha \in \Delta$ and Δ is a root basis, we must have that $\beta_i = 0$ for some i, so $\Delta \subset \Delta_\gamma$. Since they have the same cardinality, we have that $\Delta = \Delta_\gamma$. $\square$

The connected components of $E \setminus \bigcup_{\alpha \in Phi} H_\alpha$ are called *Weyl chambers.*

Lemma 6.17. *If $\gamma, \gamma' \in E \setminus \bigcup_{\alpha \in Phi} H_\alpha$ then $\Delta_\gamma = \Delta_{\gamma'}$ if and only if γ, γ' lie in the same Weyl chamber.*

Proof. γ, γ' lie in the same Weyl chamber if and only if (α, γ) and (α, γ') have the same sign for all $\alpha \in \Phi$, if and only if $\Phi_\gamma^+ = \Phi_{\gamma'}^+$, if and only if $\Delta_\gamma = \Delta_{\gamma'}$. $\square$

Thus the map $\gamma \to \Delta_\gamma$ defines a bijection between Weyl chambers and root bases, and its inverse is denoted by

$$\Delta \to \varphi(\Delta) = \{x \in E \mid (x, \alpha) > 0 \forall \alpha \in \Delta\}. \tag{6.11}$$

$\varphi(\Delta)$ is called the *fundamental Weyl chamber* attached to Δ.

Definition 6.18. If $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ is a root basis $\alpha = \sum c_i \alpha_i \in \Phi$, the *root height* of α is defined as $\sum c_i$.

Height can be large, A_n has max height n.

Lemma 6.19. *If $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ is a root basis and $\alpha \in \Phi$ is positive but not simple, then $\alpha - \alpha_i \in \Phi$ for some $\alpha_i \in \Delta$.*

Proof. If $(\alpha, \alpha_i) > 0$ for some i, then $\alpha - \alpha_i \in \Phi$ so we are done.

If $(\alpha, \alpha_i) \leq 0$ for all i, then by Claim 2, $\Delta \cup \{\alpha\}$ would consist of linearly independent elements, and this is a contradiction because Δ is a basis and $\alpha \notin \Delta$ because α is not simple. $\square$

By induction on the root height, we obtain the following corollary, which says that for any root α, we can go from 0 to α via a "path through simple roots in Φ".

Corollary 6.20. *If $\alpha \in \Phi^+$, there's a sequence $\beta_1, \dots, \beta_n$ of simple roots (not necessarily distinct) such that $\beta_1 + \cdots + \beta_n = \alpha$ and $\beta_1 + \cdots + \beta_k \in \Phi$ for all k.*

6.5 Weyl group and root bases

Let (Φ, E) be a root system with Weyl group W. If $\Delta \subset \Phi$ is a root basis, then so is $w(\Delta)$ for all $w \in W$. Moreover, W preserves the set of root hyperplanes $\{H_\alpha\}_{\alpha \in \Phi}$so W acts on the set of Weyl chambers. These actions are compatible, so that $w(\varphi(\Delta)) = \varphi(w(\Delta))$.

Theorem 6.21. *(i) W acts simply on the set of root bases, so that if Δ, Δ' are root bases, then there exists a unique $w \in W$ such that $w(\Delta) = \Delta'$.*

(ii) If $\Delta \subset \Phi$ is a fixed root basis and $\alpha \in \Phi$, then there exists $w \in W$ such that $w(\alpha) \subset \Delta$.

(iii) W is generated by simple reflections $\{w_{\alpha_i} \mid \alpha_i \in \Delta\}$ for any fixed Δ.

We will prove some lemmas before we prove this. In what follows, fix a root basis $\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subset \Phi$.

Lemma 6.22. *If $\alpha_i \in \Delta$, then w_{α_i} preserves $\Phi^+ \setminus \{\alpha_i\}$.*

Proof. See Sheet 3. □

So w_{α_i} maps $\alpha_i \to -\alpha_i$, and maps all the other elements of Φ^+ to some other element of Φ^+.

Lemma 6.23. *Set*

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \in E. \tag{6.12}$$

Then $w_{\alpha_i}(\rho) = \rho - \alpha_i$ for all $\alpha_i \in \Delta$.

Proof. This follows from Lemma 6.22. □

Proof of Theorem 6.21. Let $W' \subset W$ be the subgroup generated by simple reflections w_α for all $\alpha \in \Delta$.

Claim 1. W' acts transitively on root bases.

Indeed, it suffices to show W' acts transitively on Weyl chambers. Let $\gamma \in E \setminus \bigcup_{\alpha \in Phi} H_\alpha$. Choose $w \in W'$ such that $(w(\gamma), \rho)$ is maximal among the elements of W'. By considering $(w_{\alpha_i} w(\gamma), \rho) < (w(\gamma), \rho)$ we get that $(w(\gamma), \alpha_i) \geq 0$. Since $\gamma \notin H_\alpha$, $(w(\gamma), \alpha_i) > 0$ so $w(\gamma) \in \Delta$.

Details on the example sheet.

Claim 2. For all $\alpha \in \Phi$, there exists a $w \in W'$ such that $w(\alpha) \in \Delta$.

By Claim 1, it suffices to show α lies in some root basis of Φ. Take $\gamma \in E \setminus \bigcup_{\alpha \in Phi} H_\alpha$ such that $(\gamma, \alpha) > 0$ and $|(\gamma, \beta)| > (\gamma, \alpha)$ for all $\beta \in \Phi \setminus \{\pm\alpha\}$ (we need to show why such a γ exists). Then $\alpha \in \Delta_\gamma$. We can choose γ just off the perpendicular to α.

Claim 3. $W' = W$. Indeed, let $\alpha \in \Phi$ and let $w \in W'$ be such that $w(\alpha) \in \Delta$. Then $w_{w(\alpha)} = w \cdots w_\alpha \cdot w^{-1}$ by basic geometry, so $w_\alpha = w^{-1} \cdot w_{w(\alpha)} \cdot w \in W'$.

It remains to show that the stabilizer of Δ in W is trivial, this is done in Sheet 3. □

6.6 The Cartan matrix

Let Φ be a root system with root basis $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$.

Definition 6.24. The integers $(\alpha_i, \alpha_j^\vee)$ are called *Cartan integers* and $C = ((\alpha_i, \alpha_j^\vee))_{ij}$ is the *Cartan matrix.*

Example 6.25. 1. For A_1, $C = (2)$.

2. For A_2, $C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

3. For $A_1 \times A_1$, $C = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

4. For B_2, $C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$

5. For G_2, $C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

Proposition 6.26. *If (Φ', E') is another root system with root basis $\{\alpha_1', \dots, \alpha_\ell'\}$ and $(\alpha_i', \alpha_j'^\vee) = (\alpha_i, \alpha_j^\vee)$ for all i, j, then there exists a* unique *isomorphism $f : (\Phi, E) \to (\Phi', E')$ such that $f(\alpha_i) = \alpha_i'$.*

Proof. There exists a unique $\mathbb{R}$-linear isomorphism $f : E \to E'$ sending $\alpha \mapsto \alpha_i'$ for all i as Δ, Δ' are bases. We need to show that $f(\Phi) = \Phi'$, and $(f(\alpha), f(\beta)^\vee) = (\alpha, \beta^\vee)$. By assumption, $(f(\alpha_i), f(\alpha_j)^\vee) = (\alpha_i, \alpha_j^\vee)$. By linearity, $(f(\lambda), f(\alpha_j)^\vee) = (\lambda, \alpha_j^\vee)$ for every $\lambda \in E$. Therefore

$$w_{f(\alpha_j)}(f(\lambda)) = f(w_{\alpha_j}(\lambda)), \tag{6.13}$$

so $w_{\alpha_j} = f \circ w_{\alpha_j} \circ f^{-1}$. Consider the isomorphism

$$\begin{aligned} G : \mathrm{GL}(E) &\to \mathrm{GL}(E') \\ g &\mapsto f \circ g \circ f^{-1} \end{aligned} \tag{6.14}$$

We showed that $G(w\alpha_j) = w_{\alpha_j'}$. Since W is generated by simple reflections, G maps $W \twoheadrightarrow W'$, the Weyl group of Φ'. If $\alpha \in \Phi$, let $w \in W$ be such that $w(\alpha) = \alpha_i \in \Delta$. Let $w' = G(w) = f \circ w \circ f^{-1} \in W'$. Then $w'(f(\alpha)) = f(w(\alpha)) = f(\alpha_i) = \alpha_i'$. Then $f(\alpha) = (w')^{-1}(\alpha_i') \in \Phi'$, so $f(\Phi) = \Phi'$. Similarly, we get that $(\alpha, \beta^\vee) = (f(\alpha), f(\beta)^\vee)$ for all $\alpha, \beta \in \Phi$. ☐

Corollary 6.27. *Cartan matrices determine root systems.*

Proof. See Humphreys for a concrete construction of Φ from the Cartan matrices. ☐

Remark 6.28. Since W acts transitively on root bases, the Cartan matrix is unique up to reordering rows and columns.

6.7 Dynkin Diagrams

Let (Φ, E) be a root system, and let $\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subset \Phi$ be a root basis, and C the Cartan matrix.

Definition 6.29. The *Dynkin diagram* $D(\Phi)$ determines C, which determines Φ by the work in the previous section. It is the graph where

1. Vertices are simple roots $\alpha_1, \dots, \alpha_\ell$.

2. If $\alpha, \beta \in \Delta$, draw $(\alpha, \beta^\vee)(\beta, \alpha^\vee)$ edges between α and β recall that $(\alpha, \beta^\vee)(\beta, \alpha^\vee) \in \{0, 1, 2, 3\}$.

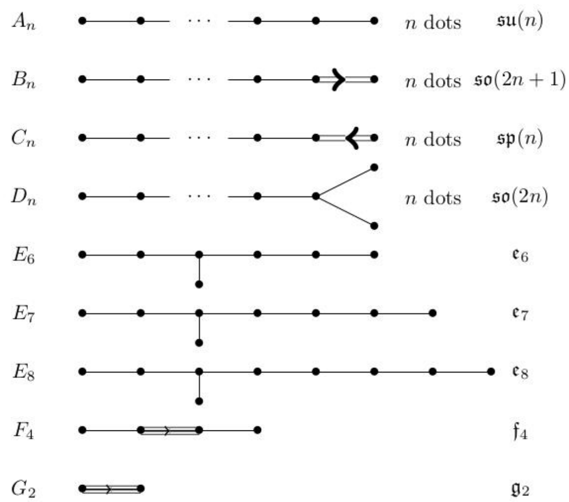

Figure 2: Dynkin diagrams

3. If $\alpha, \beta \in \Delta$ have at least one edge between them and $(\alpha, \alpha) < (\beta, \beta)$, so $(\beta, \alpha^\vee) \in \{-2, -3\}$, then draw a $<$ pointing to α.
4. Don't draw self edges from α to α.

We show some Dynkin diagrams in Figure 2. Using Euclidean geometry and combinatorics, one can show the following.

Theorem 6.30. *The association $\Phi \mapsto D(\Phi)$ induces a bijection between irreducible root systems and the Dynkin diagrams in Figure 2.*

Proof. Φ is irreducible if and only if $D(\Phi)$ is connected (Sheet 3). The rest is in Humphreys, §11.4. □

7 Root spaces: return of the Lie algebras

Summarizing our work so far, we have constructed a map from $(\mathfrak{g}, \mathfrak{t})$ a semisimple Lie algebra with Cartan subalgebra $\mathfrak{t}$ to a root system, but we don't yet know that this map is bijective. We have also constructed a bijection between root systems and disjoint unions of Dynkin diagrams. We also want to be able to "forget $\mathfrak{t}$" by showing that all CSAs are isomorphic in $\mathfrak{g}$.

7.1 Independence of $\mathfrak{t}$

Theorem 7.1 (Conjugacy of CSAs)**.** *If $\mathfrak{g}$ is a semisimple Lie algebra, and $\mathfrak{t}, \mathfrak{t}' \subset \mathfrak{g}$ are CSAs, then there exists a Lie algebra automorphism $f : \mathfrak{g} \to \mathfrak{g}$ such that $f(\mathfrak{t}) = \mathfrak{t}'$.*

Proof. Humphreys, §16.4. □

Thus is we choose $\mathfrak{t}$ and $\mathfrak{t}'$ and build root systems, they are the same.

Corollary 7.2. *Let Φ, Φ' be the root systems of $\mathfrak{g}$ in $\mathfrak{t}^*$ and $(\mathfrak{t}')^*$. Then Φ and Φ' are isomorphic as root systems.*

Proof. If $f : \mathfrak{g} \to \mathfrak{g}$ is an automorphism with $f(\mathfrak{t}) = \mathfrak{t}'$, then

$$f^* : (\mathfrak{t}')^* \to (\mathfrak{t})^* \tag{7.1}$$

is an isomorphism and preserves the pairings induced by the Killings forms, so it sends Φ' to Φ. □

Corollary 7.3. *If $\mathfrak{g}$ is semisimple with CSA $\mathfrak{t}$ giving rise to $\Phi = \Phi(\mathfrak{g}, \mathfrak{t})$, then $\mathfrak{g}$ is simple if and only if Φ is irreducible if and only if $D(\Phi)$ is connected.*

Proof. Exercise. Sketch: If $\mathfrak{g}$ is not simple, then $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Take CSAs $\mathfrak{t}_i \subset \mathfrak{g}$. Then $\mathfrak{t}_1 \oplus \mathfrak{t}_2$ is a CSA in $\mathfrak{g}$. By the conjugacy theorem, we may assume that $\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_2$, so $\Phi = \Phi_1 \sqcup \Phi_2$. □

7.2 Existence and uniqueness theorems

Theorem 7.4 (Existence)**.** *For each irreducible root system Φ, there exists a simple Lie algebra with CSA $\mathfrak{t} \subset \mathfrak{g}$ such that the root system of $(\mathfrak{g}, \mathfrak{t})$ is isomorphic to Φ.*

Theorem 7.5 (Uniqueness)**.** *Let $\mathfrak{g}, \mathfrak{g}'$ be semisimple Lie algebras with CSAs $\mathfrak{t} \subset \mathfrak{g}$, $\mathfrak{t}' \subset \mathfrak{g}'$, giving root systems Φ, Φ'. Choose root bases $\Delta \subset \Phi$, $\Delta' \subset \Phi'$. Choose, for each $\alpha \in \Delta$, a generator $\varphi_\alpha \in \mathfrak{g}_\alpha$ and similarly $\varphi'_{\alpha'} \in \mathfrak{g}'_{\alpha'}$.*

Let $f : \Phi \to \Phi'$ be an isomorphism of root systems with $f(\Delta) = \Delta'$. Then there exists a unique isomorphism $f : \mathfrak{g} \to \mathfrak{g}'$ such that $\tilde{f}(\mathfrak{t}) \subset \mathfrak{t}'$ and $\tilde{f}(e_\alpha) = \tilde{e}_{f(\alpha)}$ for all $\alpha \in \Delta$.

Proof. Sketch: Recall the root space decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha. \tag{7.2}$$

For each $\alpha \in \Phi$, choose a generator $e_\alpha \in \mathfrak{g}_\alpha$. Then we can choose e_α compatibly with e_α so that

$$[e_\alpha, e_{-\alpha}] = h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)} \in \mathfrak{t} \tag{7.3}$$

and $(e_\alpha, h_\alpha, f_\alpha)$ form an $\mathfrak{sl}_2$-triple. So $\mathfrak{g}$ has a basis

$$\{h_\alpha \mid \alpha \in \Delta\} \sqcup \{e_\alpha \mid \alpha \in \Phi\} \tag{7.4}$$

What is the Lie bracket?

- If $t \in \mathfrak{t}$, $\alpha \in \Phi$, then $[t, e_\alpha] = \alpha(t) e_\alpha$.
- If $t, t' \in \mathfrak{t}$ then $[t, t'] = 0$.
- If $\alpha \in \Phi$, then $[e_\alpha, e_{-\alpha}] = h_\alpha$.
- If $\alpha, \beta \in \Phi$ with $\alpha + \beta \notin \Phi$ and $\beta \neq \pm\alpha$, then $[e_\alpha, e_\beta] = 0$.
- If $\alpha, \beta \in \Phi$ and $\beta \neq \pm\alpha$ and $\alpha + \beta \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ so $[e_\alpha, e_\beta] = c_{\alpha\beta} e_{\alpha+\beta}$ where $c_{\alpha\beta} \neq 0$.

The theory of "structure constants": choose a nice basis of $\mathfrak{g}$ such that $c_{\alpha\beta} = \pm 1$ and the basis is "compatibly". This leads (after a lot of work) to both of these theorems. See Humphreys §18.4. $\square$

Remark 7.6. There's a slightly different approach due to Serre using generators and relations.

To conclude, we have bijections between simple Lie algebras, irreducible root systems, and the Dynkin diagrams in Figure 2.

7.3 Classical Lie algebras

Simple Lie algebras or irreducible root systems of type A_n, B_n, C_n, D_n are called *classical*. What are the classical Lie algebras?

Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $=t$ be the diagonal CSA, and $e_i : \mathfrak{t} \to \mathbb{C}$ sending a diagonal matrix to its ith entry. Then

$$\mathfrak{t}^* = (\mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_{n+1})/(e_1 + \cdots + e_n) \tag{7.5}$$

and

$$\Phi = \{e_i - e_j \mid 1 \le i \ne j \le n+1\} \tag{7.6}$$

and a root basis is

$$\Delta = \{e_1 - e_2, \ldots, e_n - e_{n+1}\} \tag{7.7}$$

By calculating root strings or otherwise, we find that

$$(\alpha_i, \alpha_j^\vee) = \begin{cases} 2 & i = j \\ -1 & |i-j| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{7.8}$$

You don't need to compute any Killing forms for this. Thus by looking at the Dynkin diagram we see that the root system is of type A_n.

The Weyl group $W \subset \mathrm{GL}(\mathfrak{t}^*)$ is generated by w_α with $\alpha = e_i - e_j$. We see that w_α swaps e_i and e_j. Since transpositions generate the symmetric group, we have that $W \cong S_{n+1}$. Similar computations with $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_n$ show that they are of the following types:

Type	$\mathfrak{g}$	$\Phi \subset \mathbb{R}^n$	Δ	W	$\dim \mathfrak{g}$
B_n	$\mathfrak{so}_{2n+1}$	$\{e_i\} \cup \{\pm e_i \pm e_j \mid i \ne j\}$	$\{e_1 - e_2, \ldots, e_{n-1} - e_n, e_n\}$	$S_n \ltimes C_2^n$	$2n^2 + n$
C_n	$\mathfrak{sp}_{2n}$	$\{2e_i\} \cup \{\pm e_i \pm e_j \mid i \ne j\}$	$\{e_1 - e_2, \ldots, e_{n-1} - e_n, 2e_n\}$	$S_n \ltimes C_2^n$	$2n^2 + n$
D_n	$\mathfrak{so}_{2n}$	$\{\pm e_i \pm e_j \mid i \ne j\}$	$\{e_1 - e_2, \ldots, e_{n-1} - e_n, e_{n-1} + e_n\}$	$S_n \ltimes C_2^{n-1}$	$2n^2 - n$

An essential exercise is to verify the claims about Φ and Δ.

Non-examinable. How does W act on Φ in each case? S_n acts on $\mathbb{R}^n$ by permuting the basis vectors. In the B_n and C_n case, $C_2^n = \{(\delta_i) \mid \delta_i = \pm 1\}$ acts via $e_i \to \delta_i e_i$. In the D_n case, $C_2^{n-1} \subset C_2^n$ with $\prod \delta_i = 1$.

Since $B_2 \cong C_2$, we have that $\mathfrak{so}_5 \cong \mathfrak{sp}_4$. Similarly, $D_3 \cong A_3$ so $\mathfrak{so}_6 \cong \mathfrak{sl}_4$. These are the "accidental isomorphisms".

7.4 Exceptional Lie algebras *for culture*

If $\mathfrak{g}$ (or Φ) has type E, F, G, we call it *exceptional*. We denote these Lie algebras by $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$. Some facts:

Type	$\#\Phi^+$	$\#W$	$\dim \mathfrak{g}$	Dim of smallest faithful rep $\mathfrak{g} \to \mathfrak{gl}_n$
G_2				

Will fill in later.

7.5 The root and weight lattice

Let (Φ, E) be a root system. A *lattice* in E is the $\mathbb{Z}$-span of an $\mathbb{R}$-basis.

Definition 7.7. The *root lattice* of Φ is

$$\mathbb{Z}\Phi = \{\sum_{\alpha \in \Phi} c_\alpha \alpha \mid c_\alpha \in \mathbb{Z}\}. \tag{7.9}$$

The *weight lattice* of Φ is

$$X = \{\alpha \in E \mid (\lambda, \alpha^\vee) \in \mathbb{Z} \forall \alpha \in \Phi\} \tag{7.10}$$

These are indeed lattices: fix a root basis $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of Φ. Then $\mathbb{Z}\Phi$ is the $\mathbb{Z}$-span of Δ, which is an $\mathbb{R}$-basis. On example sheet 3, we showed that

$$X = \{\lambda \in E \mid (\lambda, \alpha_i^\vee) \in \mathbb{Z} \forall \alpha_i \in \Delta\} \tag{7.11}$$

Let $\omega_1, \ldots, \omega_\ell \in E$ be the unique elements such that $(\omega_i, \alpha_j^\vee) = \delta_{ij}$. These are called the *fundamental weights* of Φ with respect to Δ. Clearly X is the $\mathbb{Z}$-span of $\omega_1, \ldots, \omega_\ell$.

Example 7.8. If $\Phi = \{\pm\alpha\} \cong A_1$, $\mathbb{Z}\Phi = \mathbb{Z}\alpha$, and $(\alpha, \alpha^\vee) = 2$, so $X = \mathbb{Z}(\alpha/2)$.

Example 7.9. We can do some careful calculations with A_2, B_2, G_2 and determine there weight and root lattices.

Since $(\alpha, \beta^\vee) \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$, we have that $\mathbb{Z}\Phi \subset X$. Then $X/\mathbb{Z}\Phi$ is a finite group, called the *fundamental group* of Φ. This is the fundamental group of a Lie group. Moreover,

$$\#(X/\mathbb{Z}\Phi) = |\det(C)| \tag{7.12}$$

where C is the Cartan matrix. This is somewhat intuitive.

In the previous examples, we have that the size of the fundamental group is 2 if $\Phi = A_1$, 3 if $\Phi = A_2$, 2 if $\Phi = B_2$, and 1 if $\Phi = G_2$.

Definition 7.10. An element $\lambda \in X$ is *dominant* if $(\lambda, \alpha^\vee) \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \Phi^+$. We have that $\lambda \in X$ is dominant if and only if $(\lambda, \alpha_i^\vee) \geq 0$ for all $\alpha_i \in \Delta$, if and only if $\lambda = \sum_{i=1}^{\ell} c_i w_i$ with $c_i \geq 0$, if and only if λ lies in the closure of the fundamental Weyl chamber.

8 Representations of semisimple Lie algebras

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{t} \subset \mathfrak{g}$ a CSA, and $\Phi \subset \mathfrak{t}^*$ the root system. Fix a root basis $\Delta \subset \Phi$. We have that $\mathbb{Z}\Phi \subset X \subset \mathfrak{t}^*$.

Lemma 8.1. *An element $\lambda \in \mathfrak{t}^*$ lies in X if and only if $\lambda(h_\alpha) \in \mathbb{Z}$ for all $\alpha \in \Phi$.*

Proof. Recall that $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha}$ and $\lambda(h_\alpha) = (\lambda, \alpha^\vee)$ for all $\alpha \in \Phi$. □

Now, let V be a representation of $\mathfrak{g}$. Since $\mathfrak{t}$ is abelian and every element of $\mathfrak{t}$ is semisimple as an element of $\mathfrak{g}$, we can diagonalize V under the $\mathfrak{t}$-action, so that

$$V = \bigoplus_{\lambda \in \mathfrak{t}^*} V_\lambda, \quad V_\lambda = \{v \in V \mid tv = \lambda(t)v \ \forall t \in \mathfrak{t}\} \tag{8.1}$$

Note that the Weyl group W acts on $X \subset E$.

Proposition 8.2. *(i) If $\alpha \in \Phi$ and $e_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$, then $e_\alpha \cdot V_\lambda \subset V_{\lambda+\alpha}$ (hopping to the left).*

(ii) If $V_\lambda \neq 0$, then $\lambda \in X$ (weights are integers).

(iii) $\dim V_\lambda = \dim V_{w(\lambda)}$ for all $w \in W$ (symmetry under Weyl group).

Proof. (i): This follows from direct computation: if $t \in \mathfrak{t}$ and $v \in V_\lambda$, then

$$\begin{aligned} t(e_\alpha \cdot v) &= e_\alpha(t \cdot v) + [t, e_\alpha]v \\ &= \lambda(t)(e_\alpha \cdot v) + \alpha(t)(e_\alpha v). \end{aligned} \tag{8.2}$$

(ii): Fix a root $\alpha \in \Phi$ and choose $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $\{e_\alpha, h_\alpha, f_\alpha\}$ satisfy the $\mathfrak{sl}_2$ relations, so they span $\mathfrak{m}_\alpha \cong \mathfrak{sl}_2$.

View V as a representation of $\mathfrak{m}_\alpha$. Then the h_α weights are in $\mathbb{Z}$, and these are exactly the set (compare with the definition of an $\mathfrak{sl}_2$ weight)

$$\{\lambda(h_\alpha) \mid \lambda \in \mathfrak{t}^*, \ V_\lambda \neq 0\}. \tag{8.3}$$

So $\lambda(h_\alpha) \in \mathbb{Z}$ for all $\alpha \in \Phi$ and all λ with $V_\lambda \neq 0$. So if $V_\lambda \neq 0$, then $\lambda \in X$.

(iii): We will show the result for $w = w_\alpha$, and the full result will follows because the w_αs generate W. Note that

$$(w_\alpha(\lambda))(h_\alpha) = \lambda(h_\alpha) - (\lambda, \alpha^\vee)\alpha(h_\alpha) = \lambda(h_\alpha) - \lambda(h_\alpha) \cdot 2 = -\lambda(h_\alpha). \tag{8.4}$$

Decompose V as a direct sum of irreducible $\mathfrak{m}_\alpha$ representations:

$$V = \bigoplus_{i=1}^{k} V^{(i)}. \tag{8.5}$$

Since $V^{(i)}$ is a direct sum of distinct weight spaces for h_α, V_λ has a basis $v_1, \ldots, v_n$ with the property that $v_i \in V^{(i)}$ (after reordering the summands $V^{(i)}$). This is a completely trivial fact: each $V^{(i)}$ contributes at most 1 dimension to V_λ by $\mathfrak{sl}_2$ theory.

The $-\lambda(h_\alpha)$ wieght space of $V^{(i)}$ is generated by an element of the form $f_\alpha^m \cdot v_i$ or $e_\alpha^m v_i$ for some $m \geq 0$. Therefore this element lies in $V_{w_\alpha(\lambda)}$ by (i).

□

Definition 8.3. An element $v \in V \setminus \{0\}$ is called a *highest weight vector* if $v \in V_\lambda$ for some λ, and $e_\alpha \cdot v = 0$ for all $\alpha \in \Delta$, where $e_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$. In that case λ is called a *highest weight.*

Lemma 8.4. *(i) V has a highest weight vector.*

(ii) Every highest weight vector is dominant.

Proof. (i): This follows from Proposition 8.2 (i) and the fact that $\dim_V < \infty$. Take a nonzero $v \in V_\lambda$, and apply e_α repeatedly for each $\alpha \in \Delta$. Eventually $e_\alpha v = 0$.

(ii): View V as an $\mathfrak{m}_\alpha$ representation for some $\alpha \in \Delta$. Then a highest weight vector must be one for V, viewed as an $\mathfrak{m}_\alpha$ representation. By $\mathfrak{sl}_2$ theory, $\lambda(h_\alpha) = (\lambda, \alpha^\vee) \in \mathbb{Z}_{\geq 0}$. □

8.1 The universal enveloping algebra

In this section vector spaces may be infinite dimensional, and in fact the universal enveloping algebra is always infinite dimensional for a Lie algebra $\mathfrak{g}$.

Definition 8.5. An *algebra* is a vector space A with a bilinear map $A \times A \to A$ $(x, y) \mapsto xy$. It is *unital* if it has a unit, and *associative* if $x(yz) = (xy)z$ for all $x, y, z \in A$.

Associative algebras are the mostly general structure for which it makes sense to do representation theory. If A is an associative algebra, we can define the bracket $[x, y] = xy - yx$. This defines a Lie algebra on A, we call this $\mathrm{Lie}(A)$.

Example 8.6. If V is a vector space, then $A = \mathrm{End}(V)$ is a unital associative algebra with $\mathrm{Lie}(A) = \mathfrak{gl}(V)$.

Definition 8.7. A representation of A is an algebra homomorphism

$$A \to \mathrm{End}(V). \tag{8.6}$$

If G is a finite group, we can define the group algebra $\mathbb{C}[G]$ such that representations of the finite group G correspond to $\mathbb{C}[G]$ representations.

Definition 8.8. For any Lie algebra $\mathfrak{g}$, we will construct a unital associative algebra $U(\mathfrak{g})$, called the *universal enveloping algebra* with a similar property to the group algebra $\mathbb{C}[G]$ described above.

This algebra will satisfy the following universal property: if A is a unital associative algebra and $f : \mathfrak{g} \to \mathrm{Lie}(A)$ is a Lie algebra map, there is a unique $\tilde{f} : U(\mathfrak{g}) \to A$ such that $\tilde{f} \circ i = f$. In diagram form we have

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "i"] \arrow[dr, "f"'] & U(\mathfrak{g}) \arrow[d, dashed, "\exists! \tilde{f}"] \\ & \mathrm{Lie}(A) \end{tikzcd}$$

Definition 8.9. If V is a $\mathbb{C}$-vector space, let

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n} \tag{8.7}$$

where $V^{\otimes 0} = \mathbb{C}$ and we have the graded algebra structure

$$V^{\otimes n} \times V^{\otimes m} \to V^{\otimes (n+m)} \tag{8.8}$$

by composition. This is a unital algebra called the *tensor algebra*.

Definition 8.10. A *two sided ideal* I of an algebra is a subspace I satisfying $xI \subset I$, $Ix \subset I$ for all $x \in A$.

If I is a two sided ideal, then A/I inherits the structre of an algebra.

Definition 8.11. If V is vector space, then define the *symmetric algebra* $\mathrm{Sym}(V) := T(V)/I$, where

$$I = (x \otimes y - y \otimes x \mid x, y \in V) \tag{8.9}$$

Definition 8.12. Let $\mathfrak{g}$ be a Lie algebra. Then the universal enveloping algebra is $U(\mathfrak{g}) = T(\mathfrak{g})/J$, where J is the two sided ideal generated by

$$\{x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g}\} \tag{8.10}$$

Note that this messes up the grading of our algebra, since $x \otimes y, y \otimes x \in \mathfrak{g}^{\otimes 2}$ and $[x, y] \in \mathfrak{g}$. We will write $x_1 \otimes \cdots \otimes x_n \in U(\mathfrak{g})$ as $x_1 \cdots x_n$.

Lemma 8.13. *$U(\mathfrak{g})$ satisfies the universal property of the universal enveloping algebra.*

Proof. Given a Lie algebra map $f : \mathfrak{g} \to \mathrm{Lie}(A)$, we have that f is a linear map $\mathfrak{g} \to A$. We can extend this map to $T(\mathfrak{g})$ by

$$\begin{aligned} f' : T(\mathfrak{g}) &\to A \\ x_1 \otimes \cdots \otimes x_n &\mapsto f(x_1) \cdots f(x_n) \end{aligned} \tag{8.11}$$

Since f preserves the Lie bracket, we have that $f(x \otimes y - y \otimes x - [x, y]) = 0$, so $J \subset \ker(f')$. So f' factors through an algebra map $\tilde{f} : U(\mathfrak{g}) \to A$. Let $i : \mathfrak{g} \to U(\mathfrak{g})$ be the canonical inclusion. Then $\tilde{f} \circ i = f$. Since $U(\mathfrak{g})$ is generated by $\mathfrak{g}$ as an algebra, $\tilde{f}$ must be unique. □

Applying this to $A = \mathrm{End}(V)$, we get a bijection between Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$ and algebra homomorphisms $U(\mathfrak{g}) \to \mathrm{End}(V)$. So if $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation and $x \in U(\mathfrak{g})$, then $\rho(x) = \tilde{\rho}(x)$ "makes sense". In other words, the action of $\mathfrak{g}$ on V extends to $U(\mathfrak{g})$ action on V via $xv = \tilde{\rho}(x)(v)$ for all $x \in U(\mathfrak{g})$.

Example 8.14. If $\mathfrak{g}$ is abelian, then $U(\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g})$ so $U(\mathfrak{g})$ is infinite dimensional.

Example 8.15. If $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and $\Omega = ef + fe + 1/2h^2 \in U(\mathfrak{g})$, then $\rho(\Omega) = \Omega_\rho$ is the Kasimir element. In fact, Ω lies in the center of $U(\mathfrak{g})$, so by Schur's lemma it acts by scalars.

Example 8.16. If $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, then $e(fv) = f(ev) + hv$ for any $\mathfrak{g}$-module V and any $v \in V$.

This comes from the fact that $ef = fe + h$ in $U(\mathfrak{g})$, so it carries to any representation.

8.2 Poitcare-Birkhoff-Witt Theorem (PBW)

This theorem describes bases of $U(\mathfrak{g})$. $U(\mathfrak{g})$ is not nicely graded because the ideal J is not homogeneous. But it is filtered: if F_n is the image of $\bigoplus_{k\leq n} V^{\otimes k}$ under the projection $T(\mathfrak{g}) \to U(\mathfrak{g})$, then F_n is a subspace of $U(\mathfrak{g})$ and we have a filtration

$$0 \subset F_0 \subset F_1 \subset \cdots \subset U(\mathfrak{g}), \tag{8.12}$$

and $\bigcup_{n\geq 0} F_n = U(\mathfrak{g})$ and $F_i \cdot F_j \subset F_{i+j}$. We can make a graded algebra by setting

$$\mathrm{gr}(U(\mathfrak{g})) = \bigoplus_{n\geq 0} F_n/F_{n-1} \tag{8.13}$$

where $F_{-1} = 0$. The multiplication maps $F_i \times F_j \to F_{i+j}$ induce maps

$$(F_i/F_{i-1}) \times (F_j/F_{j-1}) \to F_{i+j}/F_{i+j-1}, \tag{8.14}$$

which defines an algebra structure on $\mathrm{gr}(U(\mathfrak{g}))$.

Lemma 8.17. $\mathrm{gr}(U(\mathfrak{g}))$ *is commutative.*

Proof. Concretely, this means that the Lie bracket on $\mathrm{gr}(U(\mathfrak{g}))$ is zero. If $x, y \in \mathfrak{g}$, then $xy - yx = [x, y]$ in $U(\mathfrak{g})$, so $[F_1, F_1] \subset F_1$. Set $\mathrm{gr}^n := F_n/F_{n-1} \subset \mathrm{gr}(U(\mathfrak{g}))$. Then this shows that $xy = yx$ for all $x, y \in \mathrm{gr}^1$. Since $\mathrm{gr}(U(\mathfrak{g}))$ is generated by gr^1, $[x, y] = 0$ for all $x, y \in \mathrm{gr}(U(\mathfrak{g}))$. □

We have algebra homomorphisms

$$\begin{tikzcd}
T(\mathfrak{g}) \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, "f"] & U(\mathfrak{g}) \arrow[d] \\
\mathrm{Sym}(\mathfrak{g}) \arrow[r, "\exists\varphi"] & \mathrm{gr}(U(\mathfrak{g}))
\end{tikzcd}$$

Since $\mathrm{gr}(U(\mathfrak{g}))$ is commutative, $f(x \otimes y - y \otimes x) = 0$ for all $x, y \in \mathfrak{g}$, so f factors through φ (so the diagram above commutes).

Theorem 8.18 (PBW)**.** φ *is an isomorphism of algebras.*

Proof. We've already shown φ is surjective. To show injectivity is a bit harder, see Humphreys §17.4. □

Corollary 8.19. *If* $x_1, \dots, x_n$ *is a basis of* $\mathfrak{g}$*, then* $\{x_1^{k_1} \cdots x_n^{k_n} \mid k_j \in \mathbb{Z}_{\geq 0}\}$ *is a basis of* $U(\mathfrak{g})$*.*

Proof. This follows from the PBW theorem and the isomorphism of vector spaces

$$U(\mathfrak{g}) \cong \mathrm{gr}(U(\mathfrak{g})). \tag{8.15}$$

□

Lemma 8.20. *If* V *is a* $\mathfrak{g}$*-representation and* $v \in V$*, then*

$$U(\mathfrak{g}) \cdot v = \{xv \mid x \in U(\mathfrak{g})\} \tag{8.16}$$

is the smallest $\mathfrak{g}$ *subrepresentation of* V *containing* v*.*

Proof. If A is a unital associative algebra with representation V and $v \in V$, then $Av \subset V$ is the smallest A-subrepresentation of V containing v. We apply this to $A = U(\mathfrak{g})$ and note that $\mathfrak{g}$-subrepresentations are the same as $U(\mathfrak{g})$-subrepresentations. □

8.3 Highest weight modules

To study irreducible finite-dimensional representations, we study a larger class of representations with similar properties. Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{t} \subset \mathfrak{g}$, $\Phi \subset \mathfrak{t}^*$ the root, and $\Delta = \{\alpha_1, \ldots, \alpha_\ell\} \subset \Phi$ a root basis.

Let V be a (possibly infinite dimensional) representation of $\mathfrak{g}$. Then we can still diagonalize so that the following notion makes sense:

$$\lambda \in \mathfrak{t}^*, \quad V_\lambda = \{v \in V \mid tv = \lambda(t)v \ \forall t \in \mathfrak{t}\} \tag{8.17}$$

A *highest weight vector* is an element $v \in V_\lambda \setminus \{0\}$ for some $\lambda \in \mathfrak{t}^*$ such that $e_\alpha \cdot v = 0$ for all $\alpha \in \Phi^+$ and for all $e_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$ (equivalently for all $\alpha \in \Delta$).

Definition 8.21. *V is a highest weight module if V contains a highest weight vector $v \in V_\lambda$ and is generated by v, so that $V = U(\mathfrak{g}) \cdot v$.*

Lemma 8.22. *If V is finite dimensional and irreducible, then V is a highest weight module.*

Proof. Since V is finite dimensional, we showed that it has a highest weight vector v. By irreducibility and Lemma 8.20, we have that $V = U(\mathfrak{g}) \cdot v$. □

Thus we have shown that highest weight modules are generalizations of finite dimensional irreducible representations.

Example 8.23. If $\mathfrak{g} = \mathfrak{sl}_2$, there are infinite dimensional highest weight modules. Let V be the vector space with infinite basis $v_0, v_1, \ldots$, and $\mathfrak{g}$ action defined by

$$\begin{cases} e \cdot v_0 = 0 \\ h \cdot v_0 = 0 \\ f \cdot v_i = v_{i+1} \end{cases} \tag{8.18}$$

and extend using linearity and the Lie bracket. This is a highest weight module generated by highest weight vector v_0. But v_1 is also a highest weight vector, as we can calculate that $h \cdot v_n = -2nv_n$ and $e \cdot v_n = n(1-n)v_{n-1}$. So $U(\mathfrak{g}) \cdot v_1 = \mathrm{Span}\{v_1, v_2, \ldots\} \subset V$ is a subrepresentation, so V is not irreducible.

Definition 8.24. Define

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha. \tag{8.19}$$

These are subalgebras and as vector spaces we have that

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{t} \oplus \mathfrak{n}^-. \tag{8.20}$$

Lemma 8.25. *If V is a highest weight module generated by highest weight vector v, then $V = U(\mathfrak{n}^-) \cdot v$.*

Proof. Choose for each $\beta \in \Phi$ a nonzero element $e_\beta \in \mathfrak{g}_\beta$. Let $t_1, \ldots t_\ell \in \mathfrak{t}$ be a basis of $\mathfrak{t}$. By PBW, $U(\mathfrak{g}) \cdot v$ is spanned by elements of the form

$$e_{-\beta_1}^{a_1} \cdots e_{-\beta_n}^{a_n} t_1^{b_1} \cdots t_\ell^{b_\ell} e_{\beta_1}^{c_1} \cdots e_{\beta_n}^{c_n} \cdot v \tag{8.21}$$

where $\{\beta_1, \ldots, \beta_n\} \in \Phi^+$ and $a_i, b_i, c_i \in \mathbb{Z}_{\geq 0}$. Since v is a highest weight vector, $e_{\beta_i} \cdot v = 0$ and $t_i v = \lambda(t_i)v$, so we can assume that $b_i = c_i = 0$. □

If $\lambda, \mu \in \mathfrak{t}^*$, we say that $\mu \leq \lambda$ if $\lambda - \mu = \sum k_i \alpha_i$ with $k_i \in \mathbb{Z}_{\geq 0}$. This defines a partial order on $\mathfrak{t}^*$.

Proposition 8.26. *Let V be a highest weight module generated by highest weight vector $v_\lambda \in V_\lambda$. Then*

(i)
$$V = \bigoplus_{\mu \leq \lambda} V_\mu. \tag{8.22}$$

(ii) If $W \subset V$ is a subrepresentation, then
$$W = \bigoplus_{\mu \leq \lambda} W_\mu. \tag{8.23}$$

(iii) $\dim(V_\mu) < \infty$ *and* $\dim(V_\lambda) = 1$.

(iv) V is irreducible if and only if every highest weight vector lies in V_λ.

(v) V has a unique maximal proper subrepresentation, so there exists a unique irreducible quotient representation.

Proof. (i) By Lemma 8.25, $V = U(\mathfrak{n}^-) \cdot v_\lambda$ and is spanned by elements of the form
$$e_{-\beta_1}^{k_1} \cdots e_{-\beta_n}^{k_n} \cdot v_\lambda \tag{8.24}$$
with $k_i \in \mathbb{Z}_{\geq 0}$. By Proposition 8.2 (i), we have that $e_\alpha \cdot V_\lambda \subset V_{\lambda+\alpha}$ (even if $\dim V = \infty$), so (8.24) lies in V_μ with $\mu = \lambda - \sum k_i \beta_i$, so $\mu \leq \lambda$.

(ii) Let $W \subset V$ be a subrepresentation. We need to show that if $w \in W$ has $w = v_1 + \cdots + v_k$ with $v_i \in V_{\lambda_i}$, then $v_i \in W$ for all i. Suppose for the sake of contradiction that w is an element violating this condition, with k minimal. Then $k \geq 2$, and $v_i \notin W$ for all $1 \leq i \leq k$ by minimality. Let $t \in \mathfrak{t}$ be such that $\lambda_1(t) \neq \lambda_2(t)$. Then
$$tw - \lambda_1(t)w = (\lambda_2(t) - \lambda_1(t))v_2 + \cdots + (\lambda_k(t) - \lambda_1(t))v_k \in W \tag{8.25}$$
because W is a subrepresentation. By minimality, we have that $(\lambda_2(t) - \lambda_1(t))v_2 \in W$, so $v_2 \in W$, contradicting the fact that $v_2 \notin W$.

(iii) By Lemma 8.25, the dimension of V_μ is at most the number of tuples $k_1, \ldots, k_n$ such that $\mu = \lambda - \sum k_i \beta_i$. This is a finite set (exercise), and is 1 if $\mu = \lambda$.

(iv) If $v_\mu \subset V_\mu$ is a highest weight vector with $\mu \neq \lambda$, then $\mu \not\leq \lambda$, and $U(\mathfrak{g})v_\mu = U(\mathfrak{n}^-)v_\mu$ is a proper subrepresentation. If W is a subrepresentation and $W \neq V$, let μ be an element such that $W_\mu \neq 0$ and such that μ is maximal with respect to $\leq$. Then $w_\mu \in W_\mu \setminus \{0\}$ is a highest weight vector and $\mu \neq \lambda$.

(v): By (iv), a subrepresentation $W \subset V$ is proper if and only if $W_\lambda = 0$. So the sum of all proper subrepresentations still satisfies $W_\lambda = 0$, so it is a proper subrepresentation. Thus taking the sum of all proper subrepresentations gives a maximal proper subrepresentation. Thus V has a unique irreducible quotient.

□

Remark 8.27. Part (ii) is a special case of the general fact that the subspace of a vector space which is stable under a linear operator is the direct sum of the eigenspaces.???

8.4 Verma modules

Why do highest weight modules exist? It turns out we can construct the "biggest" highest weight module, called the *Verma module.*

Generally, if I is a left ideal of an associated algebra, then A/I has the structure of an A-representation, via $x(y+I) = xy+I$. Using this, we can construct the "biggest" highest weight module for $\lambda \in \mathfrak{t}^*$.

Definition 8.28. Let $\lambda \in \mathfrak{t}^*$, and let $J(\lambda)$ be the left ideal of $U(\mathfrak{g})$ generated by: (i) e_β for all $\beta \in \Phi^+$, where $e_\beta \in \mathfrak{g}_\beta \backslash \{0\}$ is some generator, and (ii) $t - \lambda(t) \cdot 1$ for all $t \in \mathfrak{t}$. The first condition gives that acting by e_β gives 0, and the second gives that acting by t gives $\lambda(t)$. Then $M(\lambda) = U(\mathfrak{g})/J(\lambda)$ is the *Verma module* for λ: it is a $U(\mathfrak{g})$ representation, so also a $\mathfrak{g}$-representation.

Let $m_\lambda \coloneqq 1 + J(\lambda) \in M(\lambda)$. Then $e_\beta \cdot m_\lambda = 0$ for all $\beta \in \Phi^+$ and $t \cdot m_\lambda = \lambda(t) m_\lambda$ for all $t \in \mathfrak{t}$.

Lemma 8.29. *M_λ is a highest weight module generated by $m_\lambda = 1 + J(\lambda)$. If V is another highest weight module generated by highest weight vector $v \in V_\lambda$, then there is a unique surjection of $\mathfrak{g}$-representations $\varphi : M(\lambda) \twoheadrightarrow V$ which is surjective, and maps $m_\lambda \mapsto v$ (universal property).*

Proof. We have that $e_\beta \cdot m_\lambda = 0$ for all $\beta \in \Phi^+$ and $t \cdot \mathfrak{m}_\lambda = \lambda(t)\mathfrak{m}_\lambda$ for all $t \in \mathfrak{t}$. Since 1 generates $U(\mathfrak{g})$ as a $U(\mathfrak{g})$-module, in order to show that M_λ is generated by m_λ, it suffices to show that $m_\lambda \neq 0$, so $1 \notin J(\lambda)$. This follows from the PBW Theorem 8.18 (exercise).

If V is another highest weight module with highest weight vector $v \in V_\lambda$, then we can define a map

$$\begin{aligned} \psi : U(\mathfrak{g}) &\to V \\ x &\mapsto x \cdot v \end{aligned} \tag{8.26}$$

This is a map of $\mathfrak{g}$-representations and $J(\lambda) \subset \ker \psi$, since $\psi(e_\beta) = e_\beta \cdot v = 0$ and $\psi(t - \lambda(t) \cdot 1) = tv - \lambda(t)v = 0$. So ψ factors through some $\varphi : U(\mathfrak{g})/J(\lambda) \to V$. □

Example 8.30. If $\mathfrak{g} = \mathfrak{sl}_2$, then $M(0)$ is the highest weight module given in Example 8.23. Then $\mathrm{Span}\{v_1, v_2 \ldots\}$ is a subrepresentation, with quotient congruent to $V(0)$.

By Proposition 8.26, $M(\lambda)$ has a unique maximal proper subrepresentation, so a unique irreducible quotient. Call this quotient $V(\lambda)$.

8.5 Classification of irreducible representations

It turns out that we have already classified irreducible highest weight modules.

Proposition 8.31. *The assignment $\lambda \mapsto V(\lambda)$ induces a bijection between $\mathfrak{t}^*$ and the isomorphism classes of highest weight modules.*

Proof. This is a straightforward application of Lemma 8.29 and Proposition 8.26.

If V is an irreducible highest weight module generated by highest weight vector $v \in V_\lambda$ for some $\lambda \in \mathfrak{t}^*$. Then V is a quotient of $M(\lambda)$ by Lemma 8.29. Since $M(\lambda)$ has a unique irreducible quotient, we have that $V \cong V(\lambda)$. Why is λ unique? Since V is irreducible, every highest weight vector lies in V_λ, hence is a multiple of v since $\dim V_\lambda = 1$. So if $V \cong V(\mu)$, then μ is a highest weight vector, so $\lambda = c\mu$, and the isomorphism is given by rescaling. □

Thus we have a bijection between $\mathfrak{t}^*$ and irreducible highest weight modules. We know that the finite dimensional irreducible representations are a subset of the irreducible highest weight modules, but what is the corresponding subset of $\mathfrak{t}^*$? It turns out that they have a very nice description as the intersection of the dominant weights (a cone) and the weight lattice (a lattice).

Theorem 8.32. *If $\lambda \in \mathfrak{t}^*$, then $V(\lambda)$ is finite dimensional if and only if $\lambda \in X$ and λ is dominant.*

Using this theorem, we can prove the following

Corollary 8.33. *Every finite dimensional irreducible representation V of a semisimple Lie algebra $\mathfrak{g}$ has a unique highest weight vector λ, and the assignment of V to its highest weight vector induces a bijection between finite dimensional irreducible representations of $\mathfrak{g}$ and dominant elements of the weight lattice $\lambda \in X$.*

Before we prove the theorem, we need the following lemma.

Lemma 8.34. *For each $\alpha_i \in \Delta$, let $e_i \in \mathfrak{g}_{\alpha_i}, f_i \in \mathfrak{g}_{-\alpha_i}$ and $h_i \in \mathfrak{t}$ such that (e_i, h_i, f_i) is a $\mathfrak{sl}_2$-triple. Then in $U(\mathfrak{g})$ we have for all $k \in \mathbb{Z}_{\geq 0}$ that*

$$[e_i, f_j^{k+1}] = 0, \quad i \neq j \tag{8.27}$$

$$[e_i, f_i^{k+1}] = -(k+1) f_i^k (k \cdot 1 - h_i) \tag{8.28}$$

Proof. Use induction on k. If $k = 0$, then $[e_i, f_j] \in \mathfrak{g}_{\alpha_i - \alpha_j} = \{0\}$ if $i \neq j$ because $\alpha_i, \alpha_j \in \Delta$ and $[e_i, f_i] = h_i$ by the $\mathfrak{sl}_2$-triple action.

The claim follows by induction and the identity $[A, BC] = [A, B]C + B[A, C]$. $\square$

We are now ready to prove our big theorem. For $\lambda \in X$ dominant, the key idea is to let

$$\Pi(\lambda) := \{\mu \mid V(\lambda)_\mu \neq 0\}. \tag{8.29}$$

Since $\dim(V(\lambda)_\mu) < \infty$ by Proposition 8.26, it suffices to prove that $\#\Pi(\lambda)$ is finite. We will show that $\Pi(\lambda)$ is Weyl group invariant, and that it has finitely many orbits under the action of W, so it is finite.

Proof of Theorem 8.32. If $V(\lambda)$ is finite dimensional, then we proved that $\lambda \in X$ in Proposition 8.2 (ii) and we proved that λ is dominant in Lemma 8.4 (ii).

Now let $\lambda \in X$ be dominant. We want to show that $V(\lambda)$ is finite dimensional.

Claim 1. For each i, let $n = \lambda(h_i) = (\lambda, \alpha_i^\vee)$. Then $f_i^{n+1} \cdot v = 0$.

Proof: Let $u = f^{n+1} \cdot v$. We know that $e_j \cdot v = 0$ for all j. Moreover, for all j, we have that

$$\begin{aligned} e_j \cdot u &= (f_i^{n+1} \cdot e_j) \cdot v + [e_j, f_i^{n+1}]v \\ &= [e_j, f_i^{n+1}] \cdot v \\ &= \begin{cases} 0 & j \neq i \\ -(n+1) f_i^n (nv - h_i v) & j = i. \end{cases} \end{aligned} \tag{8.30}$$

But since $h_i = nv$, we have that $e_j \cdot u = 0$ always. So if $u \neq 0$, then u would be a highest weight vector. But $u \in V_{\lambda - (n+1)\alpha_i}$, so it lies in a different weight space from v, so $u = 0$.

Claim 2: $W = \mathrm{Span}\{v, f_i v, \ldots, f_i^n v\}$ is an $\mathfrak{m}_{\alpha_i}$-subrepresentation, where $\mathfrak{m}_{\alpha_i} = \mathrm{Span}\{e_i, h_i, f_i\}$.

Proof: We need to show that W is stable under the action of e_i, h_i, f_i:

- Stable under f_i: follows from Claim 1.
- Stable under h_i: $f_i^k v \in V_{\lambda - k\alpha_i}$ so $h_i(f_i^k v) = (\lambda - k\alpha_i)(h_i) \cdot f_i^k v$.
- Stable under e_i: follows from (8.28).

Claim 3: For each i, V is a direct sum of finite-dimensional $\mathfrak{m}_{\alpha_i}$-subrepresentations of V. So every $v \in V$ lies in a finite-dimensional $\mathfrak{m}_{\alpha_i}$-subrepresentation.

Proof: Let V' be the sum of all finite dimensional $\mathfrak{m}_{\alpha_i}$-subrepresentations of V. By Claim 2, $V' \neq 0$. We need to show that V' is $\mathfrak{g}$-stable, as then $V = V'$ by irreducibility. So let $w \in V'$ be arbitrary, so that $w \in W \subset V$ for W a finite-dimensional $\mathfrak{m}_{\alpha_i}$-subrepresentation. Let

$$W' = \sum_{i=1}^{\ell} t_i \cdot W + \sum_{\beta \in \Phi} e_\beta \cdot W \tag{8.31}$$

where $t_1, \ldots, t_\ell \in \mathfrak{t}$ is a basis for $\mathfrak{t}$ and $e_\beta \in \mathfrak{g}_\beta \setminus \{0\}$ is a generator. We have that $x \cdot W \subset W'$ for all $x \in \mathfrak{g}$. Moreover, $\dim W' < \infty$ since $\dim W < \infty$. W' is also stable under $\mathfrak{m}_{\alpha_i}$: it suffices to check stability under e_i, h_i, f_i. This is an exercise, check that

$$\begin{aligned} e_i \cdot e_\beta W &\subset e_\beta \cdot e_i \cdot W + [e_i, e_\beta] \cdot W \\ &\subset e_\beta W + e_{\beta+\alpha_i} \cdot W \\ &\subset W'. \end{aligned} \tag{8.32}$$

therefore $W' \subset V'$, so $xV' \subset V'$ for all $x \in \mathfrak{g}$.

Start with arbitrary $\mathfrak{m}_{\alpha_i}$ stable W, let W' be image of W under the $\mathfrak{g}$-action, show that W' is also $\mathfrak{m}_{\alpha_i}$-stable, so $W' \subset V$, so that if $w \in V'$, then $x \cdot w \in W' \subset V'$, so $xV' \subset V'$.

Claim 4: The Weyl group W preserves $\Pi(\lambda)$, so that $\dim(V(\lambda)_\mu) = \dim(V(\lambda)_{w(\mu)}$ for all $\mu \leq \lambda$, and all $w \in W$.

Proof: We showed that this claim holds for finite dimensional representations in Proposition 8.2 (iii). The same proof works here as well.

Claim 5: The number of W orbits in $\Pi(\lambda)$ is finite.

Proof: Let $\mu \in \Pi(\lambda)$. Then μ is in the closure of some Weyl chamber. Since W acts transitively on the Weyl chambers, we may assume (after replacing μ by a W-conjugate) that $\mu \in X$ and is dominant. So each orbit has a representative in

$$S = \{\mu \in X \mid \mu \leq \lambda\}. \tag{8.33}$$

Since $X \subset \mathfrak{t}^*$ is discrete, S is a discrete subset of $\mathfrak{t}^*$. If $\mu \in S$, then $\lambda + \mu$ is dominant and $\lambda - \mu \in \mathbb{Z}_{\geq 0} \cdot \Delta$. Therefore $(\lambda + \mu, \lambda - \mu) \geq 0$, so $(\lambda, \lambda) \geq (\mu, \mu)$, so S is bounded and discrete, so S is finite.

We've shown that W acts on $\Pi(\lambda)$ with finitely many orbits. Since W is finite, $\Pi(\lambda)$ is finite, so $\dim V$ is finite. $\square$

8.6 The character of $V(\lambda)$

For $\lambda \in X$ dominant, let

$$\Pi(\lambda) = \{\mu \in X \mid \dim V(\lambda)_\mu \neq 0\} \tag{8.34}$$

viewed as a multiset of weights with multiplicity $\dim V(\lambda)_\mu$. What is $\Pi(\lambda)$? We know that

- $\Pi(\lambda) \subset \{\mu \mid \mu \leq \lambda\}$ (Proposition 8.26 (i)).
- W preserves $\Pi(\lambda)$ and the multiplicities (Claim 4 in the proof of Theorem 8.32).
- The multiplicity of μ is at most the number of ways to write $\lambda - \mu$ as a sum of positive roots (as $V(\lambda)$) is generated by highest weight vector $v \in V_\lambda$).

Proposition 8.35. *If $\mu, \lambda \in X$ are dominant, then $\mu \in \Pi(\lambda)$ if and only if $\mu \leq \lambda$.*

Proof. If $\mu \in \Pi(\lambda)$, then the claim follows from Proposition 8.26 (i).

Let $\mu \leq \lambda$ be dominant. Say that $\mu' \in X$ is *good* if $\mu \in \Pi(\lambda)$ and $\mu' = \mu + \sum k_i \alpha_i$ with $k_i \in \mathbb{Z}_{\geq 0}$. We claim that if μ' is good and $k_i > 0$ for some i, then there is a simple α_j such that $\mu' - \alpha_j$. This proves the proposition by starting with λ, which is good.

To prove the claim, since $k_i > 0$ for some i, we have that $(\sum k_i\alpha_i, \sum k_i\alpha_i) > 0$. So there exists j with $k_j > 0$ such that $(\sum k_i \alpha_i, \alpha_j) > 0$ and we have that $(\mu, \alpha_j) \geq 0$ since μ is dominant, so $(\mu', \alpha_j) > 0$. Therefore

$$\bigoplus n \in \mathbb{Z} V_{\mu' + n\alpha_j} \tag{8.35}$$

is an $\mathfrak{m}_{\alpha_j}$-representation since $(\mu', \alpha_j) > 0$, so $V_{\mu' - \alpha_j} \neq 0$, so $\mu - \alpha_j$ is good. $\square$

Note that the above Proposition does not hold if μ is not dominant.

Corollary 8.36. *$\mu \in X$ lies in $\Pi(\lambda)$ if and only if for all $w \in W$, $w(\mu) \leq \lambda$.*

Proof. Exercise. $\square$

Example 8.37. In G_2 with $\lambda = 2\omega_1$, we have that $\Pi(2\omega_1) = \{0, \omega_1, \omega_2, 2\omega_1\}$ and the conjugates.

What about the multiplicities? If $\mu \in \Pi(\lambda)$, then $\mathrm{mult}_\lambda(\mu) = \dim(V(\lambda)_\mu)$. We know that $\mathrm{mult}_\lambda(w(\lambda)) = 1$ for all $w \in W$. In general multiplicities can be greater than 1.

Example 8.38. Suppose $\mathfrak{g}$ is simple and suppose that $V = \operatorname{ad}\mathfrak{g}$ the adjoint representation. Then $V \cong V(\alpha_0)$ where α_0 is the highest root, because all the roots of the adjoint representation are roots of Φ. The weights of V are $\Phi \cup \{0\}$, and $\mathrm{mult}(\alpha) = 1$ if $\alpha \in \Phi$, but $\mathrm{mult}(0) = \dim_{\mathbb{C}} \mathfrak{t} = \operatorname{rank}\Phi = \#\Delta$.

The Weyl character formula contains information about all the multiplicities of a representation.

Definition 8.39. Let $\mathbb{Z}[X]$ be the free abelian group with generators $\{e^\lambda \mid \lambda \in X\}$. So

$$\mathbb{Z}[X] = \{\sum_{\lambda \in X} c_\lambda e^\lambda \mid c_\lambda = 0 \text{for all but finitely many } c_\lambda \in \mathbb{Z}\} \tag{8.36}$$

The assignment $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$ makes $\mathbb{Z}[X]$ a ring, called the *character ring*.

If V is a finite dimensional $\mathfrak{g}$-representation, then the *character* of V is

$$\mathrm{ch}(V) = \sum_{\lambda \in X} \dim(V_\lambda) e^\lambda \in \mathbb{Z}[X]. \tag{8.37}$$

Example 8.40. If $\mathfrak{g} = \mathfrak{sl}_2$, then $X = \mathbb{Z} \cdot (\alpha/2)$. If $t = e^{\alpha/2}$, then $\mathbb{Z}[X] = \mathbb{Z}[t, t^{-1}]$, and

$$\mathrm{ch}(V(n)) = t^n + tn - 2 + \cdots + t^{-n}. \tag{8.38}$$

In general, choosing an isomorphism $X \cong \mathbb{Z}^\ell$ determines an isomorphism $\mathbb{Z}[X] \cong \mathbb{Z}[t_1, t_1^{-1}, \ldots, t_\ell, t_\ell^{-1}]$. The following facts are using the the Weyl character formula.

1. Set
$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha. \tag{8.39}$$
We showed that if w_{α_i} is a simple reflection, then since w_{α_i} permutes $\Phi^+ \setminus \{\alpha_i\}$, $w_{\alpha_i}(\rho) = \rho - \alpha_i = \rho - (\rho, \alpha_i^\vee)\alpha_i$. So then because $(w_i, \alpha_j^\vee) = \delta_{ij}$, we have that
$$\rho = w_1 + \cdots + w_\ell, \tag{8.40}$$
where $w_1, \ldots, w_\ell$ are the fundamental weights.

2. Recall that S_n has a sign homomorphism $S_n \to \{\pm 1\}$. This generalizes to W, where $w \in W$ is interpreted as an element of the permutation group of Φ. In particular, let $w \in W$ and write $w = w_1 \cdots w_n$, where the w_is are simple reflections, and set $\mathrm{sgn}(w) = (-1)^n$. This is a well-defined (Sheet 4) group homomorphism $W \to \{\pm 1\}$.

Theorem 8.41 (Weyl character formula)**.** *If $\lambda \in X$ is dominant, then in* $\mathrm{Frac}(\mathbb{Z}[X])$ *we have that*

$$\mathrm{ch}(V(\lambda)) = \frac{\sum_{w \in W} \mathrm{sgn}(w) e^{w(\lambda+\rho)}}{e^\rho \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})} \tag{8.41}$$

Proof. Humphreys §24.3. □

We have some easy corollaries of the Weyl character formula.

Corollary 8.42 (Weyl denominator formula)**.** *In $\mathbb{Z}[X]$, we have*

$$\sum_{w \in W} \mathrm{sgn}(w) e^{w(\rho)} = e^\rho \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) \tag{8.42}$$

Proof. Take $\lambda = 0$ in the Weyl character formula. □

Corollary 8.43 (Weyl dimension formula)**.** *If $\lambda \in X$ is dominant, then*

$$\dim V(\lambda) = \prod_{\alpha \in \Phi^+} \frac{(\lambda + \rho, \alpha^\vee)}{(\rho, \alpha^\vee)} = \prod_{\alpha \in \Phi^+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)} \tag{8.43}$$

Proof. We want to set $e^\lambda = 1$ for all $\lambda \in X$, as then

$$\mathrm{ch}(V)(1) = \sum_\mu \dim V_\mu = \dim V(\lambda) \tag{8.44}$$

But the RHS of the character formula is "0/0". But we can get the identity with L'Hopital and differentiating in a clever direction (see notes). □

Example 8.44. Let $\mathfrak{g} = \mathfrak{sl}_2$. Then $\Phi = \{\pm\alpha\}$, $\rho = \alpha/2$, and $W = \{\pm 1\}$. So the Weyl character formula is

$$\operatorname{ch}(V(n\alpha/2)) = \frac{t^{n+1} - t^{-(n+1)}}{t + t^{-1}} = t^n + t^{n-2} + \cdots + t^{-n} \tag{8.45}$$

Example 8.45. Let $\mathfrak{g} = \mathfrak{sl}_3$, $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$, $\Delta = \{\alpha, \beta\}$, and $\rho = \omega_1 + \omega_2$. We can calculate

$$\dim V(n_1\omega_1 + n_2\omega_2) = \frac{(n_1+1)(n_2+1)(n_1+n_2+2)}{2} \tag{8.46}$$

Example 8.46. Let $\mathfrak{g} = \mathfrak{sp}_4 \cong C_2 \cong B_2$, and $\Delta = \{\alpha, \beta\}$ with α short. Then we can do some more calculations.