Schubert Varieties and Frobenius Splitting

Contents

1 Introduction

Let A be a ring of characteristic $p > 0$. The Frobenius morphism $F_A : A \to A$ on A is given by $F_A(a) = a^p$. If $\mathfrak{p} \in \operatorname{Spec} A$, then clearly $F_A^{-1}(\mathfrak{p}) = \mathfrak{p}$.

We can extend this notion from rings to schemes. Let k be an algebraically closed field of characteristic $p > 0$, and let X be a scheme over k. Then we can define the *Frobenius morphism* $F_X : X \to X$ on X which is the identity map on the underlying topological space and is given on sections by $F_X^{\#} : \mathcal{O}_X \to (F_X)_*\mathcal{O}_X$ sending $g \mapsto g^p$. We often abbreviate $F = F_X$ if there is no confusion about which Frobenius morphism we are taking.

The Frobenius morphism enjoys many pleasant properties. It is clearly affine, and it is finite, hence proper. Additionally, if X is a smooth variety, then F_X is flat. Given any morphism of schemes $f : X \to Y$, we have that $f \circ F_X = F_Y \circ f$. However, note that F_X is *not* a morphism of k-schemes in general. If it was, then we would have that $a^p = a$ for all $a \in k$.

If there exists a map $\varphi : F_*\mathcal{O}_X \to \mathcal{O}_X$ such that $\varphi \circ F^{\#} = \mathrm{id}_{\mathcal{O}_X}$, we say that X is *Frobenius split* (or simply *split*), and that φ is a *Frobenius splitting* (or simply a *splitting*) of X. As $F^{\#}$ is the pth-power map, we can think of a Frobenius splitting as a pth-root map. It is fairly easy to show that Frobenius split schemes have several nice properties. For instance, if X is Frobenius split, then X is reduced (Lemma 3.2), and if $\mathcal{L}$ is a line bundle on X, then $H^i(X, \mathcal{L}) = 0$ for all $i \geq 1$ (Proposition 3.9).

The goal of this essay is to show that Schubert varieties associated to a reductive group over k are Frobenius split, and derive several geometric consequences as a result. Let G be a reductive group, $B \subset G$ a Borel subgroup and $P \supset B$ a parabolic subgroup. Then the homogeneous coset space G/P is a *flag variety*, and can be given the structure of a projective variety. Letting B act on the left in G/P, the closure of a B-orbit is a *Schubert variety*; it is a closed subvariety of G/P. While Schubert varieties are not smooth in general, using Frobenius splitting we can show that they are normal and Cohen-Macaulay.

1.1 Historical context

The study of Schubert varieties has its origins in the 19th century with the *Schubert calculus*, which gives a somewhat well-founded solution to problems in enumerative geometry. For example, the Schubert calculus gives a method to count the number of lines in $\mathbb{P}^3(\mathbb{C})$ which intersect 4 given lines. While not always rigorous, the Schubert calculus inspired the development of important areas of algebraic geometry such as intersection theory. For more information on the history of the Schubert calculus and its connections to algebraic geometry, see [KL72].

In the modern context, Schubert varieties are an integral part of the theory of reductive groups, as initiated in the foundational work of Chevalley [Che05]. They have connections to areas such as combinatorics [Ful96], representation theory with the Demazure character formula [Dem74], and number theory [PRS13] in the theory of Shimura varieties. Mehta and Ramanathan [MR85] introduced the notion of Frobenius splitting and proved that Schubert varieties are Frobenius split. Using this, they gave the first proof that ample line bundles on Schubert varieties have vanishing higher cohomology. As a consequence of their Frobenius splitting, and Ramanan and Ramanathan [RR85] showed that Schubert varieties are normal and Ramanathan [Ram85] showed that they are Cohen-Macaulay (although a proof of normality without using Frobenius splitting was published by Anderson [And85] in the same year). Mehta and Srinivas [MS87] later gave a short proof of normality directly from Frobenius splitting.

1.2 Proof strategy

In order to show that a particular scheme is Frobenius split, it is often easier to prove the splitting of a larger scheme. Let $Y \subset X$ be a closed subscheme of X with ideal sheaf $\mathcal{I}_Y \subset \mathcal{O}_X$ and $\varphi : F_*\mathcal{O}_X \to \mathcal{O}_X$

a Frobenius splitting of X. Given any open set U and any $f \in \mathcal{I}_Y(U)$, we have that $f^p \in F_*\mathcal{I}_Y(U)$ as $\mathcal{I}_Y(U)$ is an ideal and $F_*\mathcal{I}_Y = \mathcal{I}_Y$ as sheaves of abelian groups. Thus $\varphi(f^p) = f$, so $\mathcal{I}_Y \subseteq \varphi(F_*\mathcal{I}_Y)$. If equality holds, we say that φ *compatibly splits* Y, and that Y is *compatibly split* in X. Importantly, if Y is compatibly split in X, then Y itself is Frobenius split (Lemma 3.4). We prove more result about compatible splitting in Section 3.1.

A key point of emphasis is the relative, local nature of Frobenius splitting. For example, if a dense open subscheme of a reduced scheme X is Frobenius split, then X itself is Frobenius split (Lemma 3.5). If $f : X \to Y$ is a morphism of varieties with X Frobenius split such that $f_*\mathcal{O}_X = \mathcal{O}_Y$, then Y is Frobenius split (Lemma 3.6). If $Y_1, Y_2 \subset X$ are both compatibly split by $\varphi : F_*\mathcal{O}_X \to \mathcal{O}_X$, then their union and intersection is compatibly split.

These results allow us to utilize duality theory to prove that Schubert varieties are Frobenius split. In Proposition 4.11, we connect the Frobenius splitting of a smooth variety to the existence of a certain section of the invertible sheaf ω_X^{-1}. This is the key result of the essay, as it allows us to give a tractable criterion for the otherwise difficult notion of Frobenius splitting. However, as Schubert varieties are not always smooth, we cannot apply this criterion directly to show that Schubert varieties are Frobenius split. Instead, we construct the Bott-Samelson-Demazure-Hansen (BSDH) variety associated to a Schubert variety, which is a smooth resolution of its singularities. By giving an explicit description of the canonical sheaf of a BSDH variety, we show that it is Frobenius split, which then implies the Frobenius splitting of the associated Schubert variety. From here, we use the properties of Frobenius split schemes to show that Schubert varieties are normal and Cohen-Macaulay.

While our criterion for splitting is very general, and Frobenius split varieties enjoy many pleasant properties generically, we frequently utilize the structure theory of reductive groups in our proofs. For example, the line bundles on Schubert varieties and BSDH varieties can be described in terms of the root system of the associated reductive group. We give some background on reductive groups in Section 2.1. We also frequently quote algebraic geometry results from the standard references [Har77, Sta25], and assume a familiarity with the subject at the level of [Har77], Chapters II and III. Our proofs are adapted from the papers of Mehta and Ramanathan [MR85], Mehta and Srinivas [MS87], and Ramanathan [Ram85], and the definitive textbook by Brion and Kumar [BK04].

1.3 Main results

Our main result is the Frobenius splitting of the Schubert varieties associated to a reductive group. We use it to prove the results which follow.

Theorem 1.1. *Let G be a reductive group over an algebraically closed field k of characteristic $p > 0$, $P \subset G$ a parabolic subgroup, and $S^P(w) \subset G/P$ a Schubert variety. Then G/P is Frobenius split, and $S^P(w)$ splits compatibly in G/P.*

We prove Theorem 1.1 in Section 5 using the machinery developed in the previous sections. The above result holds in characteristic $p > 0$ (as in characteristic 0 there is no notion of Frobenius splitting), but using semicontinuity, we can prove some results about the properties of Schubert varieties in arbitrary characteristic.

Theorem 1.2. *Let G be a reductive group over an algebraically closed field k of arbitrary characteristic, $P \subset G$ a parabolic subgroup, $S^P(w) \subset G/P$ a Schubert variety, $S^P(w') \subset S^P(w)$ a Schubert subvariety, and $\mathcal{L}$ an ample line bundle on $S^P(w)$. Then*

(i) $H^i(S^P(w), \mathcal{L}) = 0$ *for all* $i \geq 1$.

(ii) The restriction map $H^0(S^P(w), \mathcal{L}) \to H^0(S^P(w'), \mathcal{L})$ *is surjective.*

We prove Theorem 1.2 in Section 6. The result for k of characteristic $p > 0$ is true for any Frobenius split scheme (Lemma 3.9), we prove it for characteristic 0 using semicontinuity.

While Schubert varieties are not smooth in general, they are both normal and Cohen Macaulay, which means that they have only mild singularities in a certain sense. Recall that a scheme X is normal if for every $x \in X$, the local ring $\mathcal{O}_{X,x}$ is integrally closed in its field of fractions.

Theorem 1.3. *Let G be a reductive group over an algebraically closed field k of arbitrary characteristic, $P \subset G$ a parabolic subgroup, $S^P(w) \subset G/P$ a Schubert variety. Then $S^P(w)$ is normal.*

We prove Theorem 1.3 in Section 6.1. The proof relies on an inductive argument utilizing the structure theory of reductive groups as well as Theorem 1.1. Notably, there exists Frobenius split schemes which are not normal, but every Frobenius split scheme is *weakly normal* (see [BK04, §1.2]). For example, the union of two axes $\operatorname{Spec} k[x,y]/(xy)$ is Frobenius split but not normal.

We also show that all Schubert varieties are Cohen-Macaulay. If $(A, \mathfrak{m})$ is a local ring, then $a_1, \ldots, a_r \in A$ is a *regular sequence* for A if a_i is not a zero divisor in $A/(a_1, \ldots, a_{i-1})$ for all $1 \leq i \leq r$. The *depth* of A is the maximal length of a regular sequence $a_1, \ldots, a_r \in \mathfrak{m}$ contained in the maximal ideal. The ring A is *Cohen-Macaulay* if the depth of A equals its Krull dimension. A scheme X is *Cohen-Macaulay* if for every $x \in X$, $\mathcal{O}_{X,x}$ is a Cohen-Macaulay local ring. Cohen-Macaulayness can be thought of as a weak form of smoothness, as every smooth variety is Cohen-Macaulay.

Theorem 1.4. *Let G be a reductive group over an algebraically closed field k of arbitrary characteristic, $P \subset G$ a parabolic subgroup, $S^P(w) \subset G/P$ a Schubert variety. Then $S^P(w)$ is Cohen-Macaulay.*

We prove Theorem 1.4 in Section 6.2. We define the notion of a *rational resolution* $f : X \to Y$, and show that the existence of a rational resolution implies that Y is Cohen-Macaulay. In Proposition 6.7 we show that the BSDH resolutions are rational resolutions, and the proof of Theorem 1.4 follows easily. Importantly, the proof of Proposition 6.7 depends on the various properties of Frobenius split schemes developed in Section 3.

1.4 Outline of essay

In Section 2, we recall the basic theory of reductive groups and their Schubert varieties, giving plenty of examples along the way. We then construct the BSDH resolution of a Schubert variety in Section 2.3, and give a description of its canonical sheaf. In Section 3, we prove some basic facts about Frobenius split schemes, with an emphasis on the notion of compatible splitting. We also prove consequences of Frobenius splitting such as the vanishing of higher cohomology of ample line bundles. In Section 4, we develop a criterion for a smooth projective variety to be Frobenius split by using duality theory and the Cartier operator to connect Frobenius splitting to sections of the canonical sheaf. In Section 5 we apply this criterion for splitting to the BSDH varieties, and the proof of Theorem 1.1 follows. In Section 6 we derive some geometric properties of Schubert varieties from their Frobenius splitting, proving Theorems 1.2, 1.3, and 1.4. By standard semicontinuity arguments, we show that these properties hold in arbitrary characteristic. We conclude in Section 7 with a discussion of further applications of Frobenius splitting and the role of Schubert varieties in other contexts.

2 Reductive groups and Schubert varieties

In this section, we develop the theory of Schubert varieties and their standard resolutions. In Section 2.1, we recall some basic facts about linear algebraic groups. In Section 2.2, we define Schubert varieties over reductive groups and give some of their basic properties, including their connection with flag varieties.

In Section 2.3, we construct Bott-Samelson-Demazure-Hansen (BSDH) varieties, which are resolutions of singularities of Schubert varieties. The main results is Proposition 2.28, which gives a description of the canonical sheaf of BSDH varieties. We use this result in Section 5 to show that the BSDH varieties are Frobenius split, and the Frobenius splitting of Schubert varieties follows.

Throughout this section, let k be an algebraically closed field of arbitrary characteristic.

2.1 Preliminaries

In this section we recall basic results on reductive groups from the standard references [Spr98, Hum75].

For any scheme X over k and any k-algebra R, let $X(R)$ denote the R-valued points of X. Recall that if k is algebraically closed, then $X(k)$ is in canonical bijection with the closed points of X. For brevity, we often identify $X(k)$ with X.

By a *variety* over k, we mean an integral k-scheme which is separated and of finite type over k.

Definition 2.1.

1. An *algebraic group* G over k is a variety over k equipped with a multiplication morphism m and an inverse morphism i, defined as

$$\begin{aligned} m &: G \times G \to G, & (x,y) &\mapsto xy \\ i &: G \to G, & x &\mapsto x^{-1} \end{aligned} \tag{2.1}$$

   m and i are morphisms of varieties, and the induced map on k-valued points turns $G(k)$ into a group. Equivalently, an algebraic group is a group variety over k.

2. A *morphism* of algebraic groups G, G' is a morphism of varieties $f : G \to G'$ such that the induced morphism $G(k) \mapsto G'(k)$ on k-valued points is a group homomorphism.

3. If G is affine, then G is a *linear algebraic group*.

4. A *closed subgroup* of G is a closed subvariety of G which is itself an algebraic group.

5. There exists a unique irreducible component G° of G which contains the identity $e \in G(k)$. G° is a normal subgroup of finite index in G, and G is irreducible if and only if $G^\circ = G$, in which case we say that G is *connected*.

We list some important examples of linear algebraic groups.

Example 2.2.

1. The additive group: $\mathbb{G}_a = \operatorname{Spec} k[t] \cong \mathbb{A}^1_k$ has the structure of an algebraic group under the co-multiplication map

$$\begin{aligned} m^\# : k[t] &\to k[t_1, t_2] \\ t &\mapsto t_1 + t_2. \end{aligned} \tag{2.2}$$

   The co-inverse map is given by $i^\# : t \mapsto -t$. We then have that $\mathbb{G}_a(k) = (k, +)$.

2. The multiplicative group: $\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}] \cong \mathbb{A}^1_k \setminus \{0\}$ has the structure of an algebraic group under the co-multiplication map

$$\begin{aligned} m^\# : k[t, t^{-1}] &\to k[t_1, t_2, (t_1 t_2)^{-1}] \\ t &\mapsto t_1 t_2. \end{aligned} \tag{2.3}$$

   The co-inverse map is given by $i^\# : t \mapsto t^{-1}$. We then have that $\mathbb{G}_m(k) = (k^\times, \times)$.

3. The general linear group: let

$$\mathrm{GL}_n = \operatorname{Spec} k[\{t_{ij}\}_{1\le i,j\le n}, \det(t_{ij})^{-1}]. \tag{2.4}$$

Then $\mathrm{GL}_n(k)$ is the group of invertible $n\times n$ matrices, as adjoining $\det(t_{ij})^{-1}$ to $\operatorname{Spec} k[\{t_{ij}\}_{1\le i,j\le n}]$ is like ensuring that $\det(a_{ij}) \neq 0$. We have that $m^\#$ is given by matrix multiplication:

$$\begin{aligned} m^\# : k[\{t_{ij}\}] &\to k[\{r_{ij}\}, \{s_{ij}\}, \det(r_{ij})^{-1}\det(s_{ij})^{-1}] \\ t_{ij} &\mapsto \sum_{\ell=1}^{n} r_{i\ell}s_{\ell j}. \end{aligned} \tag{2.5}$$

4. The special linear group is the closed subgroup of GL_n defined by $\det(t_{ij}) = 1$. We have that

$$\mathrm{SL}_n = \operatorname{Spec} k[\{t_{ij}\}_{1\le i,j\le n}]/(\det(t_{ij}) - 1). \tag{2.6}$$

5. The diagonal group D_n is the closed subgroup of GL_n defined by $t_{ij} = 0$ if $i \neq j$. We have that

$$D_n = \operatorname{Spec} k[t_{11}, t_{11}^{-1}, \dots, t_{nn}, t_{nn}^{-1}] \cong \prod_{i=1}^{n} \mathbb{G}_m. \tag{2.7}$$

6. The group of upper triangular matrices B_n is the closed subgroup of GL_n defined by $t_{ij} = 0$ if $i < j$. The group of strictly upper triangular matrices U_n is the closed subgroup of B_n defined by $t_{ii} = 1$.

Given any linear algebraic group G, we have that G is isomorphic to a closed subgroup of GL_n for some n, which justifies the descriptor "linear".

Definition 2.3. A linear algebraic group G is *diagonalizable* if it is isomorphic to a closed subgroup of D_n. A *torus* is a connected diagonalizable group. A *maximal torus* of G is a closed subgroup $T \subseteq G$ which is a torus, and not contained in any other. For example, D_n is a maximal torus of GL_n.

Definition 2.4. An element $x \in G$ is *semisimple* if there exists a faithful representation $\rho : G \to \mathrm{GL}_n$ with $\rho(x)$ a diagonal matrix. An element $x \in G$ is *unipotent* if there exists a faithful representation $\rho : G \to \mathrm{GL}_n$ with $\rho(x)$ strictly upper triangular. G is *unipotent* if all of its elements are unipotent.

Let $(G, G) = \{ghg^{-1}h^{-1} \mid g, h \in G\}$ be the commutator subgroup. Set $G_0 = G$ and $G_i = (G_{i-1}, G_{i-1})$. We say that G is *solvable* if there exists $m \geq 0$ such that

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_m = \{e\}. \tag{2.8}$$

By the Lie-Kholchin theorem [Hum75, §17.6], any solvable subgroup can be embedded as a subgroup of B_n and any unipotent group can be embedded as a subgroup of U_n, so we can think of solvable elements as upper triangular matrices, and unipotent elements as strictly upper triangular matrices.

Definition 2.5. A *Borel subgroup* of G is a maximal connected closed solvable subgroup of G. For example, the group of upper triangular matrices B_n is a Borel subgroup of GL_n. In general, G will contain many Borel subgroups, all of which are conjugate.

Definition 2.6. The *radical* of G, denoted $R(G)$, is the unique maximal connected *normal* solvable subgroup of G. The *unipotent radical* of G, denoted $R_u(G)$, is the unique maximal connected normal unipotent subgroup of G. We have that $R_u(G)$ is the subgroup of $R(G)$ consisting of unipotent elements. If $R(G) = \{e\}$, we say that G is *semisimple*. If $R_u(G) = \{e\}$, we say that G is *reductive*.

Example 2.7. We have that $R(\mathrm{GL}_n) = k^*I$ consists of scalar matrices in GL_n. It follows that $R_u(\mathrm{GL}_n) = \{I\}$, so GL_n is reductive. Likewise, D_n is reductive and SL_n is semisimple.

For any connected G, we can give $G/R(G)$ the structure of a semisimple group and $G/R_u(G)$ the structure of a reductive group.

Definition 2.8. Let G be a reductive group and $T \subset G$ a maximal torus. The *Weyl group* W of G is $N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G. As all maximal tori are conjugate, the Weyl group is independent of the choice of T. If G is connected, then W is finite. If $w \in W$, we write $\dot{w}$ for a coset representative of w, so that $\dot{w} \in N_G(T) \subset G$.

Example 2.9. If $G = \mathrm{GL}_n$ and $T = D_n$, then $N_G(T)$ is the group of matrices of the form $P \cdot D$, where P is a permutation matrix and $D \in D_n$ is a diagonal matrix. Thus we have that $W = N_G(T)/T \cong S_n$, the symmetric group on n elements. We can uniquely represent each element of W by a permutation matrix.

Definition 2.10. A character of G is a homomorphism $\chi : G \to \mathbb{G}_m \cong k^\times$. We denote the group of characters by $X^*(G)$.

Of particular interest is the character group $X^*(T)$ for T a maximal torus. We have that $X^*(T) \cong \mathbb{Z}^n$, where n is the dimension of T.

Example 2.11. If $T = D_n$, then $D_n \cong \prod_{i=1}^n \mathbb{G}_m$, and $X^*(T) \cong \mathbb{Z}^n$ is generated by the projections $\pi_i : (x_1, \dots, x_n) \mapsto x_i$, $1 \le i \le n$, under multiplication.

Definition 2.12. To each reductive group G we can attach a root system $\Phi \subset X^*(T)$. Associated to any Borel subgroup is a set of *positive roots* $\Phi^+ \subset \Phi$, a set $\{\alpha_1, \dots, \alpha_n\} \subset \Phi^+$ of *simple roots*, and the corresponding *simple coroots* $\{\alpha_1^\vee, \dots, \alpha_n^\vee\}$.

The set of simple roots corresponds to a set of *simple reflections* $\{s_1, \dots s_n\} \subset W$. The set of simple reflections generate W. In fact, given any $w \in W$, we can write $w = s_{i_1} \cdots s_{i_r}$ as a product of simple reflections with r minimal, this is a *reduced expression* for w. We set $\ell(w) = r$ to be the *length* of w.

There exists a unique element $w_0 \in W$ of maximal length, called the *longest element*.

Example 2.13. In $G = \mathrm{GL}_3$, the Weyl group is $W \cong S_3$ generated by simple reflections $\{(12), (23)\}$. The longest element is $w_0 = (23)(12)(23) = (13)$ with length 3. It has coset representative

$$\dot{w}_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{2.9}$$

2.2 Schubert varieties

In this section, we recall the basics of Schubert varieties from [BK04, BL00]. Throughout this section, let G be a reductive group over an algebraically closed field k, $B \subseteq G$ a Borel subgroup, $T \subseteq B$ a maximal torus, and W the Weyl group.

Definition 2.14. Let G be an algebraic group. A *G-variety* is a variety X over k equipped with a morphism of varieties (the action morphism) $\theta : G \times X \to X$ such that the induced morphism $G(k) \times X(k) \to X(k)$ is a group action on $X(k)$. If $G(k)$ acts transitively on $X(k)$, then we say that X is a *homogeneous space* for G. Suppose that $G(k)$ acts transitively on $X(k)$, and let $\mathrm{Stab}_{G(k)}(x_0)$ be the stabilizer of $x_0 \in X$. Then there exists a closed subgroup $H \subseteq G$ such that $H(k) = \mathrm{Stab}_{G(k)}(x_0)$, and we can identify the left cosets $G(k)/H(k)$ with $X(k)$.

For brevity, we often identify the closed points $X(k)$ of a G-variety with X itself, and write G/H for the coset space $G(k)/H(k)$.

Example 2.15. GL_n acts on $\mathbb{G}_m$ via $m \cdot a = (\det m)a$, and the stabilizer is SL_n. Thus we have that $\mathrm{GL}_n / \mathrm{SL}_n \cong \mathbb{G}_m$.

In fact, given any closed subgroup $H \subseteq G$, we can give G/H the structure of a quasi-projective variety (so G/H is an open subvariety of a projective variety). Note that G/H is not an algebraic group in general.

Definition 2.16. A closed subgroup $P \subseteq G$ is *parabolic* if G/P is a projective variety. A parabolic subgroup P is *minimal* if the only parabolic subgroups it contains are B and itself.

We are primarily interested in the projective varieties G/P and their subvarieties. Given any reductive group G, we have that $R(G) \subset B$ for any Borel subgroup B, so $G/B \cong (G/R(G))/(B/R(G))$. Since $G/R(G)$ is semisimple, it suffices to consider all our results for G semisimple, however we state them for general reductive groups. Additionally, since all Borel subgroups are conjugate, we have that $G/B \cong G/B'$ for any two Borel subgroups $B, B' \subset G$. The following lemma characterizes the parabolic subgroups of G.

Lemma 2.17. *Let G be a connected algebraic group and let B be a Borel subgroup. Then*

(i) B is a parabolic subgroup.

(ii) A closed subgroup of G is parabolic if and only if it is contained in a Borel subgroup.

Proof. See [Hum75, §21.3]. □

To make the variety structure of G/P more concrete, we introduce the notion of a flag.

Definition 2.18. Let V be a finite dimensional vector space over k. Then the *Grassmannian* $\mathrm{Gr}_m(V)$ is the set of all m-dimensional vector spaces of V. We can give $\mathrm{Gr}_m(V)$ the structure of a projective variety via the embedding $\mathrm{Gr}_m(V) \to \mathbb{P}(\wedge^m V)$ into projective space over the mth exterior power of V.

Definition 2.19. Let V be a finite dimensional vector space over k. A *flag* is a strictly increasing chain of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_d = V. \tag{2.10}$$

A full flag is a flag with $d = \dim V$, so that $\dim V_i = i$ for all i. Let $\mathcal{F}(V)$ denote the space of full flags of V. Then $\mathcal{F}(V)$ is a projective variety via the embedding $\mathcal{F}(V) \hookrightarrow \mathrm{Gr}_1(V) \times \cdots \mathrm{Gr}_d(V)$.

Example 2.20.

(i) Let $G = \mathrm{GL}_n$ and $B = B_n$ be the group of upper triangular matrices. Then GL_n acts transitively on the space of full flags in $V = k^n$, and the stabilizer is B. Thus we have that $G/B \cong \mathcal{F}(V)$ as G-varieties.

(ii) Let $P \subset \mathrm{GL}_4 = G$ be the subgroup of 2×2 block upper triangular matrices, that is, matrices of the form

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \tag{2.11}$$

Then P is a parabolic subgroup as it contains $B = B_n$. G acts transitively on the Grassmannian $\mathrm{Gr}_2(k^4)$, and the stabilizer is P, so we have that $G/P \cong \mathrm{Gr}_2(k^4)$.

(iii) Let $Q \subset \mathrm{GL}_3 = G$ be the subgroup of matrices of the form

$$\begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix} \tag{2.12}$$

Then Q is a minimal parabolic subgroup of G. Let $W = \langle e_1, e_2 \rangle$ be the subspace spanned by the first two standard basis vectors. Then Q acts transitively on the full flags of W, and the stabilizer of this action is B, so we have that $Q/B \cong W$. But W is a 2-dimensional space, and the full flags are just 1-dimensional subspaces, so we have that $Q/B \cong \mathbb{P}(W) \cong \mathbb{P}^1_k$.

Definition 2.21. Let G be a reductive group. The *Bruhat decomposition* of G gives a partition of G into disjoint double cosets:

$$G = \bigsqcup_{w \in W} B\dot{w}B. \tag{2.13}$$

Thus we have that

$$G/B = \bigsqcup_{w \in W} B\dot{w}B/B = \bigsqcup_{w \in W} X(w) \tag{2.14}$$

where each $X(w) = B\dot{w}B/B$ is a *Bruhat cell.* Thus if we let B act on the left on G/B, each Bruhat cell is the orbit of the B-action represented by some element of $w \in W$. We have that $X(w) \cong \mathbb{A}^{\ell(w)}$ is a locally closed subscheme of G/B.

Example 2.22. Let $G = \mathrm{GL}_3$ with Weyl group $W \cong S_3$ generated by simple reflections $\{(12), (23)\}$. Let $w_\sigma \in G$ represent the permutation $\sigma \in W$. Then we have that $w_{(132)} = w_{(23)}w_{(12)}$, and the associated double cosets are the matrices in GL_3 of the form

$$X((12)) = Bw_{(12)}B = \begin{bmatrix} * & * & * \\ \square & * & * \\ 0 & 0 & * \end{bmatrix}, \qquad X((132)) = Bw_{(132)}B = \begin{bmatrix} * & * & * \\ \square & * & * \\ 0 & \square & * \end{bmatrix} \tag{2.15}$$

where we write $\square$ to denote a nonzero entry. From this it is apparent that $X(w_{(12)}) \cong \mathbb{A}^1$ and $X(w_{(132)}) \cong \mathbb{A}^2$.

Definition 2.23. A *Schubert variety* $S(w) = \overline{X(w)}$ is the closure of a Bruhat cell in G/B.

Each Schubert variety is a closed B-invariant subvariety of G/B. Likewise, any irreducible B-invariant closed subvariety is a Schubert variety. Thus a Schubert variety is a union of Bruhat cells, and the cells which appear have an explicit description in terms of the roots which are "smaller" than w.

Definition 2.24 (Bruhat order)**.** Let $w \in W$ have reduced expression $w = s_{i_1} \cdots s_{i_r}$. Then we write $x \leq w$ if x can be written "subproduct" of the reduced expression for w, so that $x = s_{i_{j_1}} \cdots s_{i_{j_m}}$ for some $1 \leq j_1 < \cdots < j_m \leq r$.

Note that the expression $x = s_{i_{j_1}} \cdots s_{i_{j_m}}$ may not be reduced. However, if $w_0 = s_{i_1} \cdots s_{i_N}$ is a reduced expression for the longest element, we have that $w \leq w_0$ for all $w \in W$, and in fact there is a reduced expression $w = s_{i_{j_1}} \cdots s_{i_{j_m}}$ for w as a subproduct of the reduced expression for w_0.

The Bruhat order allows us to give a simple decomposition of Schubert varieties into Bruhat cells:

$$S(w) = \bigsqcup_{x \leq w} X(x). \tag{2.16}$$

The *boundary* of a Schubert variety $\partial S(w)$ is the complement $S(w) \setminus X(w)$ so that

$$\partial S(w) = \bigsqcup_{x<w} X(x). \tag{2.17}$$

The boundary is the union of all the codimension 1 Schubert subvarieties of $S(w)$, so we can consider it as a Weil divisor.

Example 2.25.

(i) If $w_0 \in W$ is the long element, then $X(w_0)$ is the *big cell.* We have that $S(w_0) = \overline{X(w_0)} = G/B$.

(ii) Let $G = \mathrm{GL}_3$ and recall the setup of Example 2.22. Then $S((12)) = \overline{X((12))} = Q/B \cong \mathbb{P}^1$, where Q is given in Example 2.20 (iii).

(iii) More generally, let $s_i \in W$ be a simple reflection and set $P(s_i) \coloneqq B\dot{e}B \sqcup B\dot{s}_iB$. Then $P(s_i)$ is a minimal parabolic subgroup and $S(s_i) = P(s_i)/B \cong \mathbb{P}^1$ is a Schubert variety.

(iv) Even more generally, the parabolic subgroups of G are in bijection with subsets $I \subset \{s_1, \dots, s_n\}$ of the set of simple reflections in W. Let $W_I \subset W$ be the subgroup generated by I and let

$$P(I) \coloneqq \bigsqcup_{w \in W_I} B\dot{w}B. \tag{2.18}$$

Then for every parabolic subgroup $P \subset G$, there is a unique $I \subset \{s_1, \dots, s_n\}$ such that $P = P(I)$. The quotient $P(I)/B$ is Schubert variety, which is in fact smooth.

(v) Schubert varieties are not necessarily smooth in general: let $G = \mathrm{SL}_4$, and let $w = (13)(24) \in W$. Then $S(w)$ is not smooth (see [LS90] or [BL00, §4.5]).

We can similarly define Schubert varieties in G/P, where $P \subset G$ is any parabolic subgroup. Let $X^P(w) = BwP/P \subset G/P$ be a B-orbit in G/P. Then the closure $S^P(w) = \overline{X^P(w)}$ is a Schubert variety; it is a B-invariant closed subvariety of G/P. Conversely, any B-invariant closed subvariety of G/P is a Schubert variety. We mostly work with Schubert varieties in G/B, as we can translate our results to Schubert varieties in G/P using the natural map

$$\pi_P : G/B \to G/P, \qquad xB \mapsto xP \tag{2.19}$$

π_P is a locally trivial fibration with smooth fiber P/B, and each Schubert variety $S^P(w) \subset G/P$ is the image of $S(w)$ under the map π_P.

The following lemma from [Kem76] is useful when applying inductive arguments, as it allows us to relate a Schubert variety associated to a Weyl group element of length r with one of length $r-1$.

Lemma 2.26. *Let $w \in W$ be an element of the Weyl group, s_i a simple reflection, and $S^{P(s_i)}(w)$ the image of $S(w)$ under the projection map $\pi_{P(s_i)} : G/B \to G/P(s_i)$. Then*

(i) $\ell(ws_i) = \ell(w) - 1$ if and only if $S(w)$ is a locally trivial $\mathbb{P}^1$-fibration over $S^{P(s_i)}(w)$. In this case $\pi_{P(s_i)}(S(ws_i)) = S^{P(s_i)}(w)$.

(ii) If $\ell(ws_i) = \ell(w) + 1$ if and only if $S(w)$ is mapped birationally onto $S^{P(s_i)}(w)$, and in this case the fibers are either $\mathbb{P}^1$ or single points.

Proof. This follows from the basic facts about root systems and the Bruhat decomposition given in [Kem76, §2]. We give a sketch. The Bruhat cell $B\dot{w}B/B$ is spanned by the one parameter subgroups $U(\alpha)$ such that $w(\alpha) < 0$ where $\alpha \in \Phi$ is a simple root. Let α_i be the simple root associated to the simple reflection s_i. We have that $\ell(ws_i) = \ell(w) - 1$ if and only if $U(\alpha_i) \subset B\dot{w}B/B$, and in this case the fiber is $P(s_i)/B \cong \mathbb{P}^1$, which gives (i). The Bruhat cell $B\dot{w}B/B$ is mapped bijectively onto $B\dot{w}B/P(s_i)$ if and only if $\ell(ws_i) = \ell(w) + 1$, which gives (ii). $\square$

In order to study the structure of Schubert varieties, we study line bundles on them. Let $\lambda \in X^*(T)$ be a character of the maximal torus T. Then λ lifts uniquely to a character $\lambda \in X^*(B)$, and B acts on $\mathbb{G}_a \cong \mathbb{A}^1 \cong k$ via $b \cdot a = \lambda(b)$. This defines a 1-dimensional representation of B, which we denote k_λ. Letting B act on the product $G \times \mathbb{G}_a$ via $b \cdot (g, a) = (gb, \lambda(b)^{-1}a)$, we obtain a line bundle $(G \times k_\lambda)/B$ over G/B. By convention, we write

$$\mathcal{L}(\lambda) \coloneqq (G \times k_{-\lambda})/B. \tag{2.20}$$

Let $\chi_1, \ldots, \chi_n$ denote the fundamental weights of Φ (the dual basis for the set of simple coroots). We have that $\mathcal{L}(\chi_i) = \mathcal{O}_{G/B}(S(w_0 s_i))$ (recall that $S(w_0 s_i)$ is a prime divisor in $S(w_0) = G/B$). The decomposition of λ into a sum of fundamental weights then gives a decomposition of line bundles:

$$\lambda = \sum_{i=1}^{n} \langle \lambda, \alpha_i^\vee \rangle \chi_i \qquad \Longrightarrow \qquad \mathcal{L}(\lambda) = \bigotimes_{i=1}^{n} \mathcal{L}(\chi_i)^{\otimes \langle \lambda, \alpha_i^\vee \rangle} \tag{2.21}$$

From this, we see that the global sections $\Gamma(G/B, \mathcal{L}(\lambda)) \neq 0$ if and only if λ is dominant. Setting $\rho = \chi_1 + \cdots + \chi_n$, we have that

$$\mathcal{L}(\rho) = \bigotimes_{i=1}^{n} \mathcal{L}(\chi_i) = \mathcal{O}_{G/B}(\partial S(w_0)). \tag{2.22}$$

2.3 Bott-Samelson-Demazure-Hansen (BSDH) resolutions

Given a Schubert variety $S(w)$, we will construct a smooth resolution $Z(w)$. The material in this section is adapted from [BK04, §2.2].

For $w \in W$ with reduced expression $w = s_{i_1} \cdots s_{i_r}$, we have that

$$B\dot{w}B = (B\dot{s}_{i_1}B) \cdots (B\dot{s}_{i_r}B) \tag{2.23}$$

and in fact

$$S(w) = P(s_{i_1}) \cdots P(s_{i_r})/B. \tag{2.24}$$

Thus, even though the Schubert variety $S(w)$ may not be smooth, we may be able to construct a resolution out of the smooth varieties $P(s_i)/B \cong \mathbb{P}^1$.

Let Y and Z be algebraic groups equipped with a right and left B-action, respectively. Then we write $Y \times_B Z$ for the quotient $(Y \times Z)/B$ under the action $(y, z) \cdot b = (yb, b^{-1}z)$, if it exists. Set

$$Z(w) = P(s_{i_1}) \times_B \cdots \times_B P(s_{i_{r-1}}) \times_B (P(s_{i_r})/B) \tag{2.25}$$

Since $P(s_i)/B \cong \mathbb{P}^1$ for all i, this is an iterated $\mathbb{P}^1$-fibration, and hence a smooth variety.

We have a product map

$$\begin{aligned} \theta_w : Z(w) &\to G/B \\ [p_1, \ldots, p_r] &\mapsto p_1 \cdots p_r/B. \end{aligned} \tag{2.26}$$

It is easy to see that this map is well defined, and by the decomposition (2.24), its image is $S(w)$. Recall that $P(s_i) = B\dot{e}B \sqcup B\dot{s}_iB$. We have that $B\dot{s}_{i_1}B \times_B \cdots \times_B (B\dot{s}_{i_r}B/B)$ is an open subset of $Z(w)$, and is mapped bijectively by θ_w to $B\dot{s}_1 \cdots \dot{s}_r B = B\dot{w}B = X(w)$. The Bruhat cell $X(w)$ is open in $S(w)$, so θ_w is birational mapping $Z(w) \to S(w)$. Thus θ_w is a smooth resolution of the singularities of $S(w)$.

If $J = \{j_1, \dots j_m\} \subset \{1, \dots, r\}$ is a set of indices with $j_1 < \cdots < j_m$, set $w(J) = s_{i_{j_1}} \cdots s_{i_{j_m}}$ to be the subproduct of w. In particular, we write $\hat{j} = \{1, \dots, j-1, j+1, r\}$, so that $w(\hat{j}) = s_{i_1} \cdots \hat{s}_{i_j} \cdots s_{i_r}$ is the Weyl group element with s_{i_j} removed. We define the variety $Z(w(J))$ analogously to $Z(w)$ as

$$Z(w(J)) \coloneqq P(s_{i_{j_1}}) \times_B \cdots \times_B P(s_{i_{j_{m-1}}}) \times_B \left(P(s_{i_{j_m}})/B\right). \tag{2.27}$$

Then $Z(w(J))$ is a closed subvariety of $Z(w)$ via the embedding

$$\begin{aligned} i_{w(J)} &: Z(w(J)) \hookrightarrow Z(w) \\ [p_{j_1}, \dots, p_{j_m}] &\mapsto [1, \dots, 1, p_{j_1}, 1 \dots, 1, p_{j_m}, 1, \dots, 1] \end{aligned} \tag{2.28}$$

where p_{j_q} is mapped to the j_qth index and all the other indices are filled by 1s. Each $Z(w(\hat{j}))$ is a prime divisor in $Z(w)$, and as with Schubert varieties we can define the boundary

$$\partial Z(w) \coloneqq \bigcup_{i=1}^{r} Z(w(\hat{j})). \tag{2.29}$$

As subvarieties of $Z(w)$, we have that

$$Z(w(J)) = \bigcap_{j \notin J} Z(w(\hat{j})) \tag{2.30}$$

and in particular $\bigcap_{i=1}^{r} Z(w(\hat{j})) = \{[1, \dots, 1]\}$.

Now, let $\psi_w : Z(w) \to Z(w(\hat{r}))$ be the morphism sending $[p_1, \dots, p_r] \mapsto [p_1, \dots, p_{r-1}]$. Then ψ_w is a locally trivial $\mathbb{P}^1$-fibration, and the inclusion mapping $i_{w(\hat{r})}$ sending $[p_1, \dots, p_{r-1}] \mapsto [p_1, \dots, p_{r-1}, 1]$ is a section for ψ_w.

We want to give an explicit description of the canonical sheaf $\omega_{Z(w)}$ of $Z(w)$, so we can apply the criteria for splitting developed in Section 4. Given a line bundle $\mathcal{L}(\lambda)$ on G/B as defined in (2.20), let $\mathcal{L}_w(\lambda) = \theta_w^* \mathcal{L}(\lambda)$ be the pullback line bundle on $Z(w)$.

To prove our main result, we use the following lemma from [Ram85], which allows us to calculate $\omega_{Z(w)}$ inductively.

Lemma 2.27. *Let X, Y be smooth varieties, and let $f : X \to Y$ be a locally trivial $\mathbb{P}^1$-fibration. Let $\sigma : Y \to X$ be a section of f, and let $D = \sigma(Y)$ be the prime divisor in X. Let $\mathcal{L}$ be a line bundle on X whose degree along the fibers of f is 1. Then*

$$\omega_X \cong f^*\omega_Y \otimes \mathcal{O}_X(-D) \otimes \mathcal{L}^{-1} \otimes f^*\sigma^*\mathcal{L} \tag{2.31}$$

Proof. The key idea is to study the degree of line bundles along the fibers of f. Since f is a $\mathbb{P}^1$-fibration, the restriction of a line bundle $\mathcal{L}$ to $f^{-1}(y)$ for $y \in Y$ will be a $\mathbb{P}^1$-bundle isomorphic to $\mathcal{O}_{\mathbb{P}^1}(d)$ for some degree d. In particular, if $\mathcal{L}$ has degree 0 along the fibers of f, then $f^*\sigma^*\mathcal{L} = \mathcal{L}$.

Now, $\mathcal{O}_X(D)$ has degree 1 along the fibers of f since $D = \sigma(Y)$ and σ is a section. So $\mathcal{L} \otimes \mathcal{O}_X(-D) \cong f^*\sigma^*\mathcal{L} \otimes f^*\sigma^*\mathcal{O}_X(-D)$. By [Har77, Proposition II.8.20] we have that

$$\sigma^*\mathcal{O}_X(-D) \cong \sigma^*\omega_X \otimes \omega_Y^{-1} \tag{2.32}$$

so

$$\mathcal{L} \otimes \mathcal{O}_X(-D) \cong f^*\sigma^*\mathcal{L} \otimes f^*\sigma^*\omega_X \otimes f^*\omega_Y^{-1}. \tag{2.33}$$

Now, ω_X has degree -2 along the fibers of f as $\omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ (see [Har77, Example II.8.20.1]) and f is a $\mathbb{P}^1$-fibration. Thus $\omega_X \otimes \mathcal{L}^{\otimes 2}$ has degree 0 along the fibers, so we have that

$$f^*\sigma^*\omega_X \cong \omega_X \otimes \mathcal{L}^2 \otimes f^*\sigma^*\mathcal{L}^{-2}. \tag{2.34}$$

Substituting (2.34) into (2.33) and rearranging gives the result. □

We are now ready to prove the main result of the section.

Proposition 2.28. *We have that*

$$\omega_{Z(w)} \cong \mathcal{O}_{Z(w)}(-\partial Z(w)) \otimes \mathcal{L}_w(-\rho). \tag{2.35}$$

Proof. We proceed by induction on $\ell(w) = r$. If $r = 1$, then $Z(w) \cong P(s_{i_1})/B \cong \mathbb{P}^1$, so $\omega_{Z(w)} \cong \omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$. We have that $\partial(Z(w))$ is a single point, so $\mathcal{O}_{Z(w)}(-\partial Z(w)) = \mathcal{O}_{\mathbb{P}^1}(-1)$. The map $\theta_w : Z(w) \to G/B$ is just the inclusion $P_{i_1}/B \hookrightarrow G/B$, so by (2.22) and the fact that $P_{i_1}/B \subset \partial(S(w_0))$, we have that $\mathcal{L}_w(-\rho) \cong \mathcal{O}_{\mathbb{P}^1}(-1)$, which gives the desired result.

Now, suppose the claim holds for all $w \in W$ with $\ell(w) < r$, so that

$$\omega_{Z(w(\hat{r}))} \cong \mathcal{O}_{Z(w(\hat{r}))}(-\partial Z(w(\hat{r}))) \otimes \mathcal{L}_{w(\hat{r})}(-\rho). \tag{2.36}$$

We want to apply Lemma 2.27 for the morphism $\psi_w : Z(w) \to Z(w(\hat{r}))$ with section $i_{w(\hat{r})}$ using the line bundle $\mathcal{L}_w(\rho)$. By (2.22), we have that $\mathcal{L}_w(\rho)$ has degree 1 along the fibers of ψ_w. We have that $i_{w(\hat{r})}(Z(w(\hat{r}))) = Z(w(\hat{r})) \subset Z(w)$ is the inclusion. Since $\theta_{w(\hat{r})} = \theta_w \circ i_{w(\hat{r})}$, we have that $i^*_{w(\hat{r})}\mathcal{L}_w(\rho) = \mathcal{L}_{w(\hat{r})}(\rho)$.

Furthermore, we have that $\psi_w^*\mathcal{O}_{Z(w(\hat{r}))}(-\partial Z(w(\hat{r}))) = \mathcal{O}_{Z(w)}(-\partial Z(w) + Z(w(\hat{r})))$. Thus applying Lemma 2.27 gives

$$\begin{aligned}
\omega_{Z(w)} &= \psi_w^*\omega_{Z(w(\hat{r}))} \otimes \mathcal{O}_{Z(w)}(-Z(w(\hat{r}))) \otimes \mathcal{L}_w(-\rho) \otimes \psi_w^* i^*_{w(\hat{r})}\mathcal{L}_w(\rho) && (2.37)\\
&= \mathcal{O}_{Z(w)}(-\partial Z(w) + Z(w(\hat{r}))) \otimes \psi_w^*\mathcal{L}_{w(\hat{r})}(-\rho) \otimes \mathcal{O}_{Z(w)}(-Z(w(\hat{r}))) \otimes \mathcal{L}_w(-\rho) \otimes \psi_w^*\mathcal{L}_{w(\hat{r})}(\rho) \\
&= \mathcal{O}_{Z(w)}(-\partial Z(w)) \otimes \mathcal{L}_w(-\rho) && (2.38)
\end{aligned}$$

as desired. □

3 Frobenius splitting

In this section, we study the properties of Frobenius split schemes and compatibly split subschemes as defined in Section 1, deriving some basic consequences and introducing criteria for a variety to be Frobenius split. In Section 3.1, we introduce the notion of compatible splitting and derive some generic criteria for a scheme to be Frobenius split. In Section 3.2, we prove the key Proposition 3.9 about the cohomology of line bundles on Frobenius split schemes. The material in this section is adapted from [BK04, §1.1-1.2]. Throughout this section, k is an algebraically closed field of characteristic $p > 0$ and all schemes are defined over k.

First, recall that a scheme X is Frobenius split if $F^\# : \mathcal{O}_X \hookrightarrow F_*\mathcal{O}_X$ splits, so there exists $\varphi : F_*\mathcal{O}_X \to \mathcal{O}_X$ such that $\varphi \circ F^\# = \mathrm{id}_{\mathcal{O}_X}$. If X is split by φ, then we have a split exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{F^\#} F_*\mathcal{O}_X \xrightarrow{\varphi} \mathcal{O}_X \longrightarrow 0$$

which we refer to as the *Frobenius exact sequence.* We can think of a Frobenius splitting as a "pth-root map" on $\mathcal{O}_X$ as follows.

Since F is the identity on the underlying topological space of X, $F_*\mathcal{O}_X$ and $\mathcal{O}_X$ are equivalent as sheaves of abelian groups, but have different $\mathcal{O}_X$-module structures. Let $U \subset X$ be an open subset of X. Then for $f \in \mathcal{O}_X(U)$, $g \in F_*\mathcal{O}_X(U)$, we have that $f \cdot g = F^{\#}(f)g = f^p g$. Thus any morphism of $\mathcal{O}_X$-modules $\varphi : F_*\mathcal{O}_X \to \mathcal{O}_X$ satisfies

$$\varphi(f \cdot g) = \varphi(f^p g) = f \cdot \varphi(g) = f\varphi(g). \tag{3.1}$$

In particular, we have $\varphi(f^p) = f\varphi(1)$, so $(\varphi \circ F^{\#})(f) = f\varphi(1)$. Thus $\varphi \in \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ is a Frobenius splitting if and only if $\varphi(1) = 1$, and in fact we can check this globally. Thus we have shown the following lemma.

Lemma 3.1. *Let $\varphi \in \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ be a morphism of $\mathcal{O}_X$-modules. Then φ is a Frobenius splitting if and only if $\varphi(1) = 1$ where $1 \in \Gamma(X, \mathcal{O}_X)$ is a global section of X.*

If X is proper, so that $\Gamma(X, \mathcal{O}_X) = k$, then φ is a scalar multiple of a splitting if and only if $\varphi(1) \neq 0$. Thus we can study $\mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ in order to determine if X is Frobenius split, which we undertake in Section 4.

Frobenius split schemes enjoy many pleasant properties, which we detail in the following sections. First, we prove a simple but valuable consequence of splitting.

Lemma 3.2. *Let X be a Frobenius split scheme. Then X is reduced.*

Proof. Let φ be a splitting of X, let $U \subset X$ be an open set, and $f \in \Gamma(U, \mathcal{O}_U)$ be nilpotent. Then there exists $\nu > 0$ such that $f^{p^\nu} = 0$. But since X is Frobenius split,

$$f^{p^{\nu-1}} = (\varphi \circ F^{\#})(f^{p^{\nu-1}}) = \varphi(f^{p^\nu}) = 0. \tag{3.2}$$

By iteration, we find that $f = 0$, so X is reduced. ☐

3.1 Compatible splittings

In this section we prove more properties related to compatible Frobenius splitting. We first give the definition of compatible splitting.

Definition 3.3. Let $\varphi : F_*\mathcal{O}_X \to \mathcal{O}_X$ be a Frobenius splitting of X and let $Y \subset X$ be a closed subscheme with sheaf of ideals $\mathcal{I}_Y$. We say that φ *compatibly splits* Y, and that Y is *compatibly split* in X, if

$$\varphi(F_*\mathcal{I}_Y) \subseteq \mathcal{I}_Y. \tag{3.3}$$

We say $Y_1, \ldots, Y_m$ are *simultaneously compatibly split* if there is a single, fixed φ which compatibly splits each Y_i.

We start with an essential lemma which allows us to show in many cases that closed subschemes of a larger scheme are Frobenius split.

Lemma 3.4. *Let $Y \subset X$ be a closed subscheme. If φ compatibly splits Y, then φ induces a Frobenius splitting φ_Y of Y.*

Proof. Recall that for any affine $U \subset X$, we have that $\mathcal{O}_Y(U \cap Y) = \mathcal{O}_X(U)/\mathcal{I}_Y(U)$. Thus if $g + \mathcal{I}_Y(U) \in F_*\mathcal{O}_Y(U)$, we may define $\varphi_Y : F_*\mathcal{O}_Y \to \mathcal{O}_Y$ by

$$\varphi_Y(g + \mathcal{I}_Y(U)) = \varphi(g) + \mathcal{I}_Y(U) \tag{3.4}$$

and since $\varphi(F_*\mathcal{I}_Y) = \mathcal{I}_Y$, we have that φ_Y is well defined. Further, we have that

$$\begin{aligned} \varphi_Y \circ F^{\#}(g + \mathcal{I}_Y(U)) &= \varphi_Y(g^p + \mathcal{I}_Y(U)) \\ &= g + \mathcal{I}_Y(U) \end{aligned} \tag{3.5}$$

so φ_Y is a Frobenius splitting of Y. □

In particular, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I}_Y & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_Y & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle F^{\#}} & & \downarrow{\scriptstyle F^{\#}} & & \downarrow{\scriptstyle F^{\#}} & & \\
0 & \longrightarrow & F_*\mathcal{I}_Y & \longrightarrow & F_*\mathcal{O}_X & \longrightarrow & F_*\mathcal{O}_Y & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_Y} & & \\
0 & \longrightarrow & \mathcal{I}_Y & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_Y & \longrightarrow & 0
\end{array}$$

with φ, φ_Y Frobenius splittings of X and Y, respectively.

Compatible splittings have several nice properties. In particular, the following lemma tells us it is sufficient to check compatible splitting on a dense open subset.

Lemma 3.5. *Let X be a reduced scheme of finite type and $U \subset X$ a dense open subset. Let $\varphi \in \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ be a morphism such that $\varphi|_U$ is a Frobenius splitting of U. Then*

(i) φ is a Frobenius splitting of X.

(ii) If $Y \subset X$ is a reduced closed subscheme such that $U \cap Y$ is dense in Y and $\varphi|_U$ compatibly splits $U \cap Y$, then φ compatibly splits Y.

Proof. (i): Since X is reduced and $\varphi(1)|_U = 1$, we have that $\varphi(1) = 1$ since U is dense in X. Thus φ is a Frobenius splitting by Lemma 3.1.

(ii): We have that $\varphi(F_*\mathcal{I}_Y)$ is coherent and $\mathcal{I}_Y \subset \varphi(F_*\mathcal{I}_Y)$, so $\varphi(F_*\mathcal{I}_Y) = \mathcal{I}_Z$ for some closed subscheme $Z \subset Y \subset X$. Now, since $U \cap Y$ is compatibly split, we have that $\mathcal{I}_Y|_{U\cap Y} = \mathcal{I}_Z|_{U \cap Y}$. We can consider Z as a closed subscheme of Y, with associated ideal sheaf $\mathcal{I}_{Z/Y}$. Then $\mathcal{I}_{Z/Y}|_{U\cap Y} = 0$, so $\mathcal{I}_{Z/Y} = 0$, so $Z = Y$, so $\varphi(F_*\mathcal{I}_Y) = \mathcal{I}_Y$, so φ compatibly splits Y.

□

Instead of directly proving that Schubert varieties $S^P(w)$ are Frobenius split, we instead show that the BSDH varieties $Z(w)$ are split, and use the morphisms $\theta_w : Z(w) \to S(w)$ and $\pi_P : S(w) \to S^P(w)$ to relate the splitting of $Z(w)$ with that of $S^P(w)$. This is made possible with the following lemma.

Lemma 3.6. *Let $f : X \to Y$ be a quasicompact morphism of reduced schemes such that $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism. Let $i : Z \hookrightarrow X$ be a closed subscheme and let $W = f(Z)$ denote the scheme-theoretic image of Z in Y. Then*

(i) Let $\mathcal{I}_Z, \mathcal{I}_W$ denote the ideal sheaves of Z, W respectively. Then $\mathcal{I}_W = f_*\mathcal{I}_Z$.

(ii) If X is Frobenius split, then Y is Frobenius split.

(iii) If Z is compatibly split in X, then W is compatibly split in Y.

Proof. (i): Since f is quasicompact, we have that (see [Sta25, Lemma 01R8])

$$\mathcal{I}_W = \ker\left((f \circ i)^{\#} : \mathcal{O}_Y \to f_* i_* \mathcal{O}_Z\right). \tag{3.6}$$

Since $\mathcal{O}_Y \cong f_*\mathcal{O}_X$, we have that

$$\begin{aligned}\mathcal{I}_W &= \ker\left(f_*\mathcal{O}_X \to f_* i_* \mathcal{O}_Z\right)\\ &= f_* \ker\left(\mathcal{O}_X \to i_*\mathcal{O}_Z\right)\\ &= f_*\mathcal{I}_Z.\end{aligned} \tag{3.7}$$

(ii): Let $\varphi : F_*\mathcal{O}_X \to \mathcal{O}_X$ be a Frobenius splitting of X, and consider the direct image

$$f_*\varphi : f_*F_*\mathcal{O}_X \to f_*\mathcal{O}_X. \tag{3.8}$$

Since $f_*F_*\mathcal{O}_X = F_*f_*\mathcal{O}_X$ and $f_*\mathcal{O}_X \cong \mathcal{O}_Y$, we have that $f_*\varphi$ is a map $F_*\mathcal{O}_Y \to \mathcal{O}_Y$, and since φ is a Frobenius splitting we have that $f_*\varphi(1) = 1$, so $f_*\varphi$ is a Frobenius splitting of Y by Lemma 3.1.
(iii): If Z is compatibly split by φ, then $\varphi(F_*\mathcal{I}_Z) = \mathcal{I}_Z$, and by part (i) $\mathcal{I}_W = f_*\mathcal{I}_Z$, so we have that

$$\begin{aligned}f_*\varphi(F_*\mathcal{I}_W) &= f_*\varphi(f_*F_*\mathcal{I}_Z)\\ &= f_*(\varphi F_*\mathcal{I}_Z)\\ &= \mathcal{I}_W\end{aligned} \tag{3.9}$$

so W is compatibly split by $f_*\varphi$. ☐

Lemma 3.7. *Let $Y, Z \subset X$ be closed subschemes of X which are simultaneously compatibly split. Then $Y \cap Z$ and $Y \cup Z$ are compatibly split.*

Proof. Let $\mathcal{I}_Y, \mathcal{I}_Z, \mathcal{I}_{Y\cap Z}$ be the ideal sheaves of $Y, Z, Y \cap Z$, respectively. Then $\mathcal{I}_{Y\cap Z} = \mathcal{I}_Y + \mathcal{I}_Z$ and $\mathcal{I}_{Y\cup Z} = \mathcal{I}_Y \cap \mathcal{I}_Z$. Let φ be a Frobenius splitting of X compatible with Y and Z. Then

$$\varphi(F_*\mathcal{I}_{Y\cap Z}) = \varphi(F_*(\mathcal{I}_Y + \mathcal{I}_Z)) = \varphi(F_*(\mathcal{I}_Y)) + \varphi(F_*(\mathcal{I}_Z)) = \mathcal{I}_{Y\cap Z} \tag{3.10}$$

as desired. The result for $Y \cup Z$ follows similarly. ☐

3.2 Line bundles on Frobenius split schemes

We now derive some nice properties of line bundles on Frobenius split schemes.

Lemma 3.8. *Let $\mathcal{L}$ be an invertible sheaf on a scheme X. Then*

(i) $F^*\mathcal{L} \cong \mathcal{L}^p$

(ii) $F_*(F^*\mathcal{L}) \cong \mathcal{L} \otimes_{\mathcal{O}_X} F_*\mathcal{O}_X$

Proof. (i): Since F is the identity on X, we have that $F^*\mathcal{L} \cong \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X$, where $\mathcal{O}_X$ acts on itself via the pth power map $f \to f^p$. Thus we have a natural $\mathcal{O}_X$-linear isomorphism $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X \to \mathcal{L}^p$ given by $\ell \otimes f \to f\ell^p$.

(ii): This follows immediately from the projection formula [Har77, Exercise III.8.3]. ☐

Proposition 3.9. *Let X be a proper Frobenius split scheme over k, and let $\mathcal{L}$ be an ample line bundle on X. Then*

(i) $H^i(X, \mathcal{L}) = 0$ *for all* $i \geq 1$.

(ii) If $Y \subset X$ *is compatibly split, then the restriction map* $H^0(X, \mathcal{L}) \to H^0(Y, \mathcal{L})$ *is surjective.*

Proof. (i): Let φ be a splitting of X. Tensoring the Frobenius exact sequence by $\mathcal{L}$ gives a split exact sequence

$$0 \longrightarrow \mathcal{L} \xrightarrow{\mathrm{id} \otimes F^{\#}} \mathcal{L} \otimes F_* \mathcal{O}_X \xrightarrow{\mathrm{id} \otimes \varphi} \mathcal{L} \longrightarrow 0.$$

Since the above sequence splits, the map on cohomology is injective:

$$H^i(X, \mathcal{L}) \hookrightarrow H^i(X, \mathcal{L} \otimes F_* \mathcal{O}_X). \tag{3.11}$$

But we have that

$$H^i(X, \mathcal{L} \otimes F_* \mathcal{O}_X) \cong H^i(X, F_*(\mathcal{L}^p)) \tag{3.12}$$

by Lemma 3.8, and since F is affine, we have an isomorphism of sheaves of abelian groups (which is *not* a k-morphism) $H^i(X, F_*(\mathcal{L}^p)) \cong H^i(X, \mathcal{L}^p)$. Thus we have an injection $H^i(X, \mathcal{L}) \hookrightarrow H^i(X, \mathcal{L}^p)$, and iterating gives an injection

$$H^i(X, \mathcal{L}) \hookrightarrow H^i(X, \mathcal{L}^{p^\nu}) \tag{3.13}$$

for all ν. But since $\mathcal{L}$ is ample, by Serre's vanishing theorem [Har77, Proposition III.5.3], there exists $n_0 \in \mathbb{Z}_{>0}$ such that $H^i(X, \mathcal{L}^n) = 0$ for all $n \geq n_0$. Thus for ν sufficiently large $H^i(X, \mathcal{L})$ injects into 0, so $H^i(X, \mathcal{L}) = 0$.

(ii): Let $Y \subset X$ be a closed subvariety compatibly split by φ. As with the Frobenius exact sequence, iterating the splitting φ gives a split exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{(F^{\#})^\nu} F^\nu_* \mathcal{O}_X \xrightarrow{\varphi^\nu} \mathcal{O}_X \longrightarrow 0$$

Tensoring by $\mathcal{L}$ and taking global sections, we get a commutative diagram with surjective horizontal maps

$$\begin{array}{ccc} H^0(X, \mathcal{L}^{p^\nu}) & \xrightarrow{\varphi^\nu} & H^0(X, \mathcal{L}) \\ \downarrow \scriptstyle{\mathrm{res}} & & \downarrow \scriptstyle{\mathrm{res}} \\ H^0(Y, \mathcal{L}^{p^\nu}) & \xrightarrow{\varphi^\nu} & H^0(Y, \mathcal{L}) \end{array}$$

We want to show that the restriction map on the right is surjective, and it suffices to show that the restriction map on the left is surjective. We have an exact sequence

$$0 \longrightarrow \mathcal{L}^{p^\nu} \otimes \mathcal{I}_Y \longrightarrow \mathcal{L}^{p^\nu} \longrightarrow \mathcal{L}^{p^\nu}|_Y \longrightarrow 0$$

and taking cohomology gives an exact sequence

$$H^0(X, \mathcal{L}^{p^\nu}) \longrightarrow H^0(Y, \mathcal{L}^{p^\nu}) \longrightarrow H^1(X, \mathcal{L}^{p^\nu} \otimes \mathcal{I}_Y)$$

Since $\mathcal{L}$ is ample, by Serre's vanishing theorem, $H^1(X, \mathcal{L}^{p^\nu} \otimes \mathcal{I}_Y) = 0$ for large ν, so $H^0(X, \mathcal{L}^{p^\nu}) \to H^0(Y, \mathcal{L}^{p^\nu})$ is surjective, so we are done. □

4 Criteria for splitting

In this section, we prove a criterion for a projective variety to be Frobenius split. The key result is Proposition 4.11, which we use to show that Schubert varieties are Frobenius split in Section 5. Throughout this section, k is an algebraically closed field of characteristic $p > 0$ and all schemes are defined over k. The material in this section is adapted from [BK04, §1.3].

In Section 4.1, we study the De Rham complex of a regular affine k-algebra. In Section 4.2, we "sheafify" the results of the previous section, proving results about the De Rham complex of a smooth k-variety and defining the trace map. In Section 4.3, we relate the trace map to Frobenius splitting, proving our splitting criterion Proposition 4.11.

By Lemma 3.1, in order to determine if a scheme X is Frobenius split, it suffices to understand the *evaluation map*:

$$\begin{aligned} \epsilon : \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) &\to \Gamma(X, \mathcal{O}_X) \\ \varphi &\mapsto \varphi(1). \end{aligned} \tag{4.1}$$

We can "sheafify" this map:

$$\begin{aligned} \epsilon : \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) &\to \mathcal{O}_X \\ \varphi &\mapsto \varphi(1) \end{aligned} \tag{4.2}$$

In order to get a handle on the evaluation map, we utilize duality theory. In particular, we will construct an isomorphism

$$\mathcal{H}om_{\mathcal{O}_X}(\omega_X, F_*\omega_X) \to \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \tag{4.3}$$

where ω_X is the canonical sheaf, defined below. We could give such an isomorphism non-explicitly using duality theory (see [Sta25, §0ATZ] or [Har77, §III.8], for instance), but our explicit (if long-winded) approach will allows us to give a concrete criterion for Frobenius splitting.

Now, let X/k be a smooth variety of dimension n. Then the Kahler differential $\Omega^1_{X/k}$ is a locally free $\mathcal{O}_X$-module of rank n, equipped with a k-derivation $d : \mathcal{O}_X \to \Omega^1_{X/k}$. See [Har77, §II.8] for more information on the Kahler differential. We can define the De Rham complex of X over k

$$\Omega^\bullet_{X/k} = \bigoplus_{i=0}^{\infty} \wedge^i \Omega^1_{X/k}, \tag{4.4}$$

which is an associative $\mathcal{O}_X$-algebra with ith graded part $\Omega^i_{X/k} = \wedge^i \Omega^1_{X/k}$ and $\Omega^0_{X/k} = \mathcal{O}_X$. In particular, we have that $\Omega^n_{X/k} = \omega_{X/k} = \omega_X$ is the *canonical sheaf* of X. As $\Omega^1_{X/k}$ is locally free of rank n, ω_X will be an invertible sheaf on X.

4.1 The affine De Rham complex

To better understand the De Rham complex, we first work locally over some open affine $\operatorname{Spec} A \subset X$, where A is a k-algebra. Then we can define the Kahler differential $\Omega^1_{A/k}$ of A, which is an A-module spanned by the symbols $\{da \mid a \in A\}$ subject to the relations $d(a+b) = da+db$ and $d(ab) = a(db)+b(da)$. $\Omega^1_{A/k}$ is equipped with a k-linear derivation $d = d_{A/k} : A \to \Omega^1_{A/k}$ sending $a \mapsto da$. Since we work in characteristic p, we have that $da^p = pa^{p-1}da = 0$, so d is A^p-linear. This fact is essential and allows us to develop our subsequent results.

The *De Rham complex* of A is

$$\Omega^\bullet_{A/k} = \bigoplus_{i=0}^{\infty} \wedge^i \Omega_{A/k}, \tag{4.5}$$

which is an associative algebra with product given by $\wedge$ and additional relations $da \wedge da = 0$ and $da \wedge db = -db \wedge da$ for $a, b \in A$. We can naturally define the ith graded part of $\Omega^\bullet_{A/k}$ to be $\Omega^i_{A/k} = \wedge^i \Omega_{A/k}$ (and set $\Omega^0_{A/k} = A$), so we have that

$$\alpha \wedge \beta = (-1)^{ij} \beta \wedge \alpha \tag{4.6}$$

for $\alpha \in \Omega^i_{A/k}$ and $\beta \in \Omega^j_{A/k}$. Any graded ring satisfying (4.6) is said to be *graded-commutative*, so $\Omega^\bullet_{A/k}$ is a graded-commutative A-algebra.

The derivation d extends to a map $d : \Omega^\bullet_{A/k} \to \Omega^\bullet_{A/k}$ given by

$$d(a_0 da_1 \wedge \cdots \wedge da_i) = da_0 \wedge da_1 \wedge \cdots \wedge da_i \tag{4.7}$$

In particular, we have that d maps $\Omega^i_{A/k} \to \Omega^{i+1}_{A/k}$ and $d^2 = 0$, so we can define a cohomology complex $H^\bullet_{A/k}$ as follows.

Let

$$Z^\bullet_{A/k} \coloneqq \{\alpha \in \Omega^\bullet_{A/k} \mid d\alpha = 0\} = \ker\left(d : \Omega^\bullet_{A/k} \to \Omega^\bullet_{A/k}\right) \tag{4.8}$$

be the complex of "cocycles" of $\Omega^\bullet_{A/k}$. Let

$$B^\bullet_{A/k} \coloneqq \{d\alpha \mid \alpha \in \Omega^\bullet_{A/k}\} = \operatorname{im}\left(d : \Omega^\bullet_{A/k} \to \Omega^\bullet_{A/k}\right) \tag{4.9}$$

be the complex of "coboundaries". Since d is A^p-linear, we have that $Z^\bullet_{A/k}$ is a graded A^p-subalgebra of $\Omega^\bullet_{A/k}$ as if $\alpha \in Z^\bullet_{A/k}$ and $a^p \in A^p$, then $d(a^p\alpha) = a^p d\alpha = 0$ so $a^p\alpha \in Z^\bullet_{A/k}$. Likewise, $B^\bullet_{A/k}$ is a graded ideal of $Z^\bullet_{A/k}$ as if $d\alpha \in B^\bullet_{A/k}$ and $\beta \in Z^\bullet_{A/k}$, then $\beta d\alpha = d(\alpha\beta)$ so $\beta d\alpha \in B^\bullet_{A/k}$.

It follows that we can define the cohomology complex

$$H^\bullet_{A/k} \coloneqq Z^\bullet_{A/k} / B^\bullet_{A/k} \tag{4.10}$$

which will be a grade-commutative A^p-algebra.

Lemma 4.1. *If A is the localization of a finitely generated k-algebra, then $\Omega^\bullet_{A/k}$ is a finitely generated A-module and $Z^\bullet_{A/k}$, $B^\bullet_{A/k}$, $H^\bullet_{A/k}$ are finitely generated A^p-modules.*

Proof. Let A' be a finitely generated k-algebra, so that $A' = k[t_1, \ldots, t_n]/I$ for some ideal I. If A is a localization of A', then $\Omega^1_{A/k}$ is generated by $dt_1, \ldots, dt_n$ and it follows that $\Omega^\bullet_{A/k}$ is a finitely generated A-module. Since A is a finitely generated A^p-module $\Omega^\bullet_{A/k}$ is finitely generated as an A^p-module, as are the submodules $Z^\bullet_{A/k}$ and $B^\bullet_{A/k}$, and the quotient module $H^\bullet_{A/k}$. ☐

We will construct a homomorphism $\Omega^\bullet_{A/k} \to H^\bullet_{A/k}$ by defining a k-derivation $A \to H^1_{A/k}$ and using the universal property of $\Omega^\bullet_{A/k}$.

Lemma 4.2. *Let $\delta : A \to H^1_{A/k}$ be given by $\delta(a) = a^{p-1}da + B^1_{A/k}$. Then δ is a well-defined k-derivation, where A is given the standard A-module structure, and A acts on $H^1_{A/k}$ via the Frobenius morphism $F : A \to A^p$.*

Proof. First, we have that $a^{p-1}da \in Z^\bullet_{A/k}$ so δ is a well-defined map.

Next, we have that

$$\begin{aligned} \delta(ab) &= a^{p-1}b^{p-1}d(ab) + B^1_{A/k} \\ &= a^p b^{p-1} db + b^p a^{p-1} da + B^1_{A/k} \\ &= a \cdot \delta(a) + b \cdot \delta(b). \end{aligned} \tag{4.11}$$

Lastly, we need to show that $\delta(a+b) = \delta(a)+\delta(b)$. Since $\frac{1}{p}\binom{p}{i}$ is not divisible by p for all $1 \leq i \leq p-1$, we have

$$\frac{1}{p}\binom{p}{i} d(a^i b^{p-i}) = \binom{p-1}{i-1} a^{i-1} b^{p-i} da + \binom{p-1}{i} a^i b^{p-i-1} db. \tag{4.12}$$

Summing over all $1 \leq i \leq p-1$ and reindexing gives

$$\begin{aligned}\sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} d(a^i b^{p-i}) &= \sum_{j=0}^{p-2} \binom{p-1}{j} a^j b^{p-1-j} da + \sum_{i=1}^{p-1} \binom{p-1}{i} a^i b^{p-i-1} db \\ &= (a+b)^{p-1} d(a+b) - a^{p-1} da - b^{p-1} db. \end{aligned} \tag{4.13}$$

The left hand side lies in $B^1_{A/k}$, so modding out gives $\delta(a+b) - \delta(a) - \delta(b) = 0$ as desired, so δ is a k-derivation.

□

By the universal property of Kahler differentials, δ induces an A-linear map

$$\begin{aligned}\gamma : \Omega^1_{A/k} &\to H^1_{A/k} \\ adb &\mapsto a \cdot \delta(b) + B^1_{A/k} = a^p b^{p-1} db + B^1_{A/k}. \end{aligned} \tag{4.14}$$

We can extend this to a map on the complexes given by

$$\begin{aligned}\gamma : \Omega^\bullet_{A/k} &\to H^\bullet_{A/k} \\ a_0 da_1 \wedge \cdots \wedge da_r &\mapsto a_0^p (a_1^{p-1} da_1) \wedge \cdots \wedge (a_r^{p-1} da_r) \mod B^\bullet_{A/k} \end{aligned} \tag{4.15}$$

where A acts on $H^1_{A/k}$ via $F : A \to A^p$.

If A is regular (so if the underlying variety is smooth), then γ is an isomorphism. This stems from the fact that in characteristic p the derivation d is A^p-linear. Thinking of γ as a sort of pth power map, the next theorem then tells us that every element of the cohomology complex can be represented uniquely by a pth power.

Proposition 4.3. *If A is regular, then $\gamma : \Omega^\bullet_{A/k} \to H^\bullet_{A/k}$ is an isomorphism.*

Proof. It suffices to prove the isomorphism locally, and as A is regular, we may assume that $A = k[[t_1, \ldots, t_n]]$ (see Definition 4.5 below).

Let $\alpha \in Z^i_{A/k}$, so that $d\alpha = 0$. We can decompose α as

$$\alpha = \sum_{j=0}^{\infty} t_n^j (\alpha_j + \beta_j \wedge dt_n) \tag{4.16}$$

where $\alpha_j \in \Omega^i_{k[[t_1,\ldots,t_{n-1}]]/k}$ and $\beta_j \in \Omega^{i-1}_{k[[t_1,\ldots,t_{n-1}]]/k}$. Then taking the differential of both sides gives that

$$\sum_{j=0}^{\infty} \left(j(-1)^i t_n^{j-1} \alpha_j \wedge dt_n + t_n^j d\alpha_j + t_n^j d\beta_j \wedge dt_n \right) = 0. \tag{4.17}$$

Comparing coefficients gives $d\alpha_j = 0$ and $(j+1)(-i)^i \alpha_{j+1} + d\beta_j = 0$. Now, we have that

$$d(t_n^{j+1} \beta_j) = (j+1)(-1)^{i+1} t_n^j \beta_j \wedge dt_n + t_n^{j+1} \wedge d\beta_j, \tag{4.18}$$

so if $p \nmid (j+1)$, we have that

$$t_n^{j+1}\alpha_{j+1} + t_n^j \beta_j \wedge dt_n = \frac{(-1)^{i+1}}{j+1} d(t_n^{j+1}\beta_j) \in B_{A/k}^{i-1}. \tag{4.19}$$

It follows that we can represent α in $H_{A/k}^i$ by terms with $\alpha_j = 0$ and $\beta_{j-1} = 0$ unless $p \mid j$. But in this case, we have that $d\alpha_j = d\beta_j = 0$, so $\alpha_j \in Z^i_{k[[t_1,\ldots,t_{n-1}]]/k}$ and $\beta_j \in Z^{i-1}_{k[[t_1,\ldots,t_{n-1}]]/k}$. Thus, repeating the above argument for each α_j, β_j and iterating, we see that α can be represented in $H^i_{A/k}$ by

$$\alpha' = \sum_{\mathbf{j}=(j_1,\ldots,j_n)} \sum_{\substack{\boldsymbol{\ell}=(\ell_1,\ldots,\ell_i) \\ 1\le \ell_1 < \cdots < \ell_i \le n}} c_{\mathbf{j},\boldsymbol{\ell}} (t_1^{j_1}\cdots t_n^{j_n})^p (t_{\ell_1}^{p-1} dt_{\ell_1}) \wedge \cdots \wedge (t_{\ell_i}^{p-1} dt_{\ell_i}). \tag{4.20}$$

This representation is unique, as $\alpha' \in B^i_{A/k}$ if and only if $c_{\mathbf{j},\boldsymbol{\ell}} = 0$ for all $\mathbf{j}, \boldsymbol{\ell}$. Furthermore, we have that

$$\gamma(t_1^{j_1}\cdots t_n^{j_1} dt_{\ell_1} \wedge \cdots \wedge dt_{\ell_i}) = (t_1^{j_1}\cdots t_n^{j_n})^p (t_{\ell_1}^{p-1} dt_{\ell_1}) \wedge \cdots \wedge (t_{\ell_i}^{p-1} dt_{\ell_i}) \tag{4.21}$$

so γ is an isomorphism. □

4.2 Sheafification

Let us return to the situation where X/k is a smooth variety of dimension n. First, note that $\Omega^\bullet_{X/k}$ is a graded-commutative $\mathcal{O}_X$-algebra. As in the affine case, we can extend the derivation $d : \mathcal{O}_X \to \Omega^1_{X/k}$ to a derivation $d : \Omega^\bullet_{X/k} \to \Omega^\bullet_{X/k}$. We have that $d : F_*\Omega^\bullet_{X/k} \to F_*\Omega^\bullet_{X/k}$ is $\mathcal{O}_X$-linear, which is analogous to the property that the derivation on $\Omega^\bullet_{A/k}$ is A^p-linear.

Likewise, we can define cohomology sheaves $\mathcal{H}^i F_*\Omega^\bullet_{X/k}$ which are locally equivalent to $H^i\Omega^\bullet_{A/k}$, and by Lemma 4.1, $\mathcal{H}^i F_*\Omega^\bullet_{X/k}$ is a coherent $\mathcal{O}_X$-module. We can take a direct sum to define

$$\mathcal{H}^\bullet F_*\Omega^\bullet_{X/k} := \bigoplus_{i=0}^{\infty} \mathcal{H}^i F_*\Omega^\bullet_{X/k} \tag{4.22}$$

which will be a graded-commutative $\mathcal{O}_X$-algebra. Then, by Lemma 4.2, we obtain a homomorphism

$$\gamma : \Omega^1_{X/k} \to \mathcal{H}^1 F_*\Omega^\bullet_{X/k} \tag{4.23}$$

given by $\gamma(f dg) = f^p g^{p-1} dg \mod d\mathcal{O}_X$. This extends uniquely to a homomorphism of graded-commutative $\mathcal{O}_X$-algebras

$$\gamma : \Omega^\bullet_{X/k} \to \mathcal{H}^\bullet_{X/k}. \tag{4.24}$$

Since X is a smooth variety, γ is an isomorphism by Proposition 4.3. The *Cartier operator* is the inverse isomorphism $C = \gamma^{-1}$. Since C is an isomorphism of graded-commutative $\mathcal{O}_X$-algebras, it induces isomorphisms of each graded part:

$$C_i : \mathcal{H}^i F_*\Omega^\bullet_{X/k} \to \Omega^i_{X/k}. \tag{4.25}$$

Recall the canonical sheaf $\omega_X = \Omega^n_{X/k}$ and note that since $\Omega_{X/k}$ is locally free of rank n, $\Omega^i_{X/k} = 0$ for $i > n$ so $\mathcal{H}^n F_*\Omega^\bullet_{X/k} = F_*\left(\Omega^n_{X/k}/d\Omega^{n-1}_{X/k}\right)$. Thus we can define the projection

$$\pi_n : F_*\omega_X = F_*\Omega^n_{X/k} \twoheadrightarrow F_*\left(\Omega^n_{X/k}/d\Omega^{n-1}_{X/k}\right) = \mathcal{H}^n F_*\Omega^\bullet_{X/k}. \tag{4.26}$$

Definition 4.4. The *trace map* $\tau : F_*\omega_X \to \omega_X$ is given by $C_n \circ \pi_n$.

The isomorphism γ as defined in (4.15) can be seen as a sort of pth power map, so we can think of the trace map as a pth root map on ω_X. We make this precise in the next lemma, which describes the trace map locally.

First, we give some background on systems of local coordinates and completion of local rings from [Har77, §II.9] and [Vak24, §28.1]. In particular, we describe how to embed the stalk $\mathcal{O}_{X,x}$ of a smooth variety into a power series ring. This greatly simplifies subsequent calculations. We also introduce multi-index notation. Given $\mathbf{i} = (i_1, \dots, i_n) \in \mathbb{Q}^n$, we write $t^{\mathbf{i}} = t_1^{i_1} \cdots t_n^{i_n}$, and if $a \in \mathbb{Q}$, then we write $\mathbf{a} = (a, \dots, a) \in \mathbb{Q}^n$. We also write $\mathbf{dt} = dt_1 \wedge \cdots \wedge dt_n$.

Definition 4.5. Now, let $x \in X$ be a closed point of a smooth variety X. By a *system of local coordinates* for x, we mean a minimal set of generators $t_1, \dots, t_n$ for the maximal ideal $\mathfrak{m} \subset \mathcal{O}_{X,x}$, or equivalently, a basis for $\mathfrak{m}/\mathfrak{m}^2$ as a $\mathcal{O}_{X,x}/\mathfrak{m}$-vector space. Let $U = \operatorname{Spec} A \subseteq X$ be an affine open of X containing x, and let $\mathfrak{p} \in \operatorname{Spec} A$ be the prime ideal corresponding to x. Then we can interpret $t_1, \dots, t_n$ as elements of the localization $A_{\mathfrak{p}}$, and there exists an open neighborhood $V \subset \operatorname{Spec} A$ containing x such that for each $\mathfrak{q} \in V$, $t_1, \dots, t_n \in A_{\mathfrak{q}}$ and $t_1, \dots, t_n$ are a system of local coordinates for $\mathfrak{q}$ (hence the name *local coordinates*).

If $t_1, \dots, t_n$ is a system of local coordinates for x, then $dt_1, \dots, dt_n$ is a basis for $\Omega_{X/k,x}$, and $\omega_{X,x}$ is spanned by $dt_1 \wedge \cdots \wedge dt_n$. Let $\widehat{\mathcal{O}_{X,x}}$ denote the completion of $\mathcal{O}_{X,x}$ at the ideal $\mathfrak{m}$. We have that $\widehat{\mathcal{O}_{X,x}} \cong k[x_1, \dots, x_n]$, with the injection $\mathcal{O}_{X,x} \hookrightarrow k[x_1, \dots, x_n]$ given by $t_i \mapsto x_i$ and $k \to k$.

Lemma 4.6. *Let X be a smooth variety of dimension n. Choose some closed point $x \in X$ and let $t_1, \dots, t_n$ be a system of local coordinates for x at X. Then the trace map τ at x is given locally by*

$$\tau(f\mathbf{dt}) = \operatorname{Tr}(f)\mathbf{dt} \tag{4.27}$$

where

$$\operatorname{Tr}\left(\sum_{\mathbf{i}} a_{\mathbf{i}} t^{\mathbf{i}}\right) := \sum_{\mathbf{i}} b_{\mathbf{i}} t^{(\mathbf{i}-\mathbf{p}+\mathbf{1})/p}, \qquad b_{\mathbf{i}} = \begin{cases} a_{\mathbf{i}}^{1/p} & (\mathbf{i}-\mathbf{p}+\mathbf{1})/p \in \mathbb{Z}^n \\ 0 & \textit{otherwise} \end{cases} \tag{4.28}$$

Proof. Since ω_X is locally free, we have that $\omega_{X,x} \cong \mathcal{O}_{X,x}$ so $\widehat{\omega_{X,x}} \cong k[[t_1, \dots, t_n]]$ via the mapping $f\mathbf{dt} \mapsto f$. We then have that

$$\widehat{\omega_{X,x}/d\Omega^{n-1}_{X/k,x}} \cong k[[t_1, \dots, t_n]]/I \tag{4.29}$$

where $I \subset k[[t_1, \dots, t_n]]$ is the ideal consisting of all partial derivatives of formal power series. This is true because if $f \in k[[t_1, \dots, t_n]]$, then

$$d(f dt_1 \wedge \cdots \wedge \widehat{dt_i} \wedge \cdots \wedge dt_n) = \frac{\partial f}{\partial dt_i}\mathbf{dt} \tag{4.30}$$

where $\widehat{dt_i}$ means we skip that particular index. Given $g \in k[[t_1, \dots, t_n]]$, we have that $g = \frac{\partial f}{\partial t_i}$ for some $f \in k[[t_1, \dots, t_n]]$ if and only if

$$g = \sum_{\mathbf{i}} a_{\mathbf{i}} t^{\mathbf{i}} \tag{4.31}$$

with $a_{\mathbf{i}} = 0$ if $(\mathbf{i} + \mathbf{1})/p \in \mathbb{Z}^n$. Thus we can represent any element $g + I \in k[[t_1, \dots, t_n]]/I$ by

$$\sum_{\mathbf{j}} a_{\mathbf{j}} t^{\mathbf{p}-\mathbf{1}+p\mathbf{j}} = t^{\mathbf{p}-\mathbf{1}} \left(\sum_{\mathbf{j}} a_{\mathbf{j}} t^{\mathbf{j}}\right)^p = t^{\mathbf{p}-\mathbf{1}} f^p. \tag{4.32}$$

where $f = \sum_{\mathbf{j}} a_{\mathbf{j}} t^{\mathbf{j}}$. By (4.15), we have that $\gamma(f\mathbf{dt}) = t^{\mathbf{p}-\mathbf{1}} f^p \mathbf{dt}$, so we are done as the maps $\mathbf{i} \mapsto (\mathbf{i} - \mathbf{p} + \mathbf{1})/p$ and $\mathbf{j} \mapsto (\mathbf{p} - \mathbf{1} + p\mathbf{j})$ are inverses of each other. □

4.3 The trace map and Frobenius splitting

Lemma 4.6 shows that the trace map looks like a pth root map on ω_X, which gives us hope that we may be able to relate it to Frobenius splittings of X, which themselves look like pth root maps. Our explicit description of the trace allows us to give an explicit isomorphism $\mathcal{H}om_{\mathcal{O}_X}(\omega_X, F_*\omega_X) \cong \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ which is compatible with the evaluation map defined in (4.2). Lemma 3.1 tells us that understanding the evaluation map allows us determine the Frobenius splitting of X. Thus we can use our explicit isomorphism to relate the Frobenius splitting of X to the existence of certain sections of the canonical sheaf ω_X, which we do in Proposition 4.11.

It follows from (4.28) that $\mathrm{Tr} \in \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ since $\mathrm{Tr}(f^p g) = f\,\mathrm{Tr}(g)$. In fact, $\mathrm{Tr}(f)$ is a generator for $\mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ as a $F_*\mathcal{O}_X$-module under the action $(g \cdot \mathrm{Tr})(f) = \mathrm{Tr}(fg)$ as any element of $\mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ is uniquely determined by its values on the monomials $t^{\mathbf{i}}$ with $\mathbf{i} \le \mathbf{p-1}$. Explicitly, if $\varphi \in \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$, then setting

$$h = \sum_{\mathbf{i} \le \mathbf{p-1}} \varphi(t^{\mathbf{i}}) t^{\mathbf{p-1-i}}, \tag{4.33}$$

we have that $\varphi = h \cdot \mathrm{Tr}$.

Lemma 4.7. *Let X be a smooth variety. We have an explicit isomorphism of $F_*\mathcal{O}_X$-modules*

$$\iota : \mathcal{H}om_{\mathcal{O}_X}(\omega_X, F_*\omega_X) \to \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \tag{4.34}$$

given locally at $x \in X$ with local coordinates $t_1, \dots, t_n$ by the equality in ω_X

$$\iota(\psi)(f)\mathbf{dt} = \tau(f\psi(\mathbf{dt})) \tag{4.35}$$

where $\psi \in \mathcal{H}om_{\mathcal{O}_X}(\omega_X, F_*\omega_X)_x$, $f \in \mathcal{O}_{X,x}$. Further, defining $\hat{\tau} : \mathcal{H}om_{\mathcal{O}_X}(\omega_X, F_*\omega_X) \to \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \omega_X)$ by $\hat{\tau}(\psi) = \tau \circ \psi$, we have that the diagram

$$\begin{array}{ccc} \mathcal{H}om_{\mathcal{O}_X}(\omega_X, F_*\omega_X) & \xrightarrow{\iota} & \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \\ \downarrow{\scriptstyle \hat{\tau}} & & \downarrow{\scriptstyle \epsilon} \\ \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \omega_X) & \xrightarrow{\sim} & \mathcal{O}_X \end{array}$$

commutes where ϵ is the evaluation homomorphism defined in (4.2).

Proof. First we show that ι is well-defined. Let $\psi \in \mathcal{H}om_{\mathcal{O}_X}(\omega_X, F_*\omega_X)_x$ and set $\psi(\mathbf{d}t) = g\mathbf{dt}$. Then $\iota(\psi)(f) = \mathrm{Tr}(fg)$, so $\iota(\psi) \in \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)_x$.

If $\mathbf{s} = (s_1, \dots, s_n)$ is another set of local coordinates for x, then there exists $u \in \mathcal{O}_{X,x}^{\times}$ such that $\mathbf{ds} = u\mathbf{dt}$. If $\iota(\psi)(f)\mathbf{dt} = \tau(f\psi(\mathbf{dt}))$, then

$$\iota(\psi)(f)\mathbf{ds} = u\tau(f\psi(\mathbf{dt})) = \tau(u^p f\psi(\mathbf{dt})) = \tau(f\psi(u\mathbf{dt})) = \tau(f\psi\mathbf{ds}). \tag{4.36}$$

If $g \in \mathcal{O}_{X,x}$, we have that

$$\iota(g\psi)(f)\mathbf{dt} = \tau(f(g\psi)(\mathbf{dt})) = \tau(fg\psi(\mathbf{dt})) = \iota(\psi)(fg)\mathbf{dt} \tag{4.37}$$

so ι is $F_*\mathcal{O}_X$-linear. Additionally, we have that $\epsilon(i(\psi)) = \iota(\psi)(1) = \iota(\psi)(1)\mathbf{dt} = \tau(\psi(\mathbf{dt}))$ so the diagram commutes.

In order to show that ι is an isomorphism, we need to show that it maps a generator for $\mathcal{H}om_{\mathcal{O}_X}(\omega_X, F_*\omega_X)_x$ to a generator for $\mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)_x$. We can define a generator ψ_0 for $\mathcal{H}om_{\mathcal{O}_X}(\omega_X, F_*\omega_X)_x$ by setting $\psi_0(\mathbf{dt}) = \mathbf{dt}$, and then extending linearly to get $\psi_0(f\mathbf{dt}) = f^p\mathbf{dt}$. We then have that $\iota(\psi_0) = \mathrm{Tr}$, and Tr is a generator of $\mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)_x$ so we are done. □

The properties of the Frobenius morphism allows us to restate the previous lemma in a pleasant way.

Lemma 4.8. *We have an isomorphism* $\hat{\imath} : F_*(\omega_X^{1-p}) \to \mathcal{H}om_{\mathcal{O}_X}(F^*\mathcal{O}_X, \mathcal{O}_X)$ *which gives an isomorphism of global sections* $\hat{\imath} : H^0(X, \omega_X^{1-p}) \to \mathrm{Hom}_{\mathcal{O}_X}(F^*\mathcal{O}_X, \mathcal{O}_X)$.

Proof. By the adjunction of F_* and F^*, we have that

$$\mathcal{H}om_{\mathcal{O}_X}(\omega_X, F_*\omega_X) \cong F_*\mathcal{H}om_{\mathcal{O}_X}(F^*\omega_X, \omega_X). \tag{4.38}$$

By Lemma 3.8, we have that

$$\mathcal{H}om_{\mathcal{O}_X}(F^*\omega_X, \omega_X) \cong \mathcal{H}om_{\mathcal{O}_X}(\omega_X^p, \omega_X) \cong \mathcal{H}om_{\mathcal{O}_X}(\omega_X^{p-1}, \mathcal{O}_X) \cong \omega_X^{1-p} \tag{4.39}$$

which gives the result after noting that $H^0(X, \omega_X^{1-p}) \cong H^0(X, F_*(\omega_X)^{1-p})$ as sheaves of abelian groups. □

Thus we have finally characterized $\mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ as promised at the beginning of the section. We say that $\varphi \in H^0(X, \omega_X^{1-p})$ *splits* X if its image $\hat{\imath}(\varphi)$ is a Frobenius splitting of X. The next result allows us to characterize Frobenius splitting using the trace map τ.

Lemma 4.9. *Let* X *be a smooth variety. Then*

(i) $\varphi \in H^0(X, \omega_X^{1-p})$ *splits* X *if and only if* $\hat{\tau}(\varphi) = 1$.

(ii) If X *is proper, then* φ *splits* X *if and only if at some closed point* $x \in X$ *with local coordinates* $t_1, \dots, t_n$, *we have that*

$$\varphi_x = \sum_{\mathbf{i}} a_{\mathbf{i}} t^{\mathbf{i}} \tag{4.40}$$

with $a_{\mathbf{p-1}} = 1$.

Proof. (i): By Lemma 3.1, φ splits X if and only if $\epsilon(\hat{\imath}(\varphi)) = 1$. By Lemma 4.7, we have that $\epsilon(\hat{\imath}(\varphi)) = \hat{\tau}(\varphi)$. Let $x \in X$ and $\varphi_x = f(\mathbf{dt})^{1-p}$ for some $f \in \mathcal{O}_{X,x}$ and some system of local coordinates $t_1, \dots, t_n$. By Proposition 4.6, we have locally around x that $\hat{\tau}(\varphi) = \mathrm{Tr}(f)$. If X is proper, then the global functions on X are constant, so this means we have $\hat{\tau}(f(\mathbf{dt})^{1-p}) = \mathrm{Tr}(f) \in k$ globally, so $\varphi = ct^{\mathbf{p-1}}$ and $\mathrm{Tr}(f) = c$, so φ is a splitting if and only if $c = 1$. □

The above proposition tells us that in order to prove that a proper smooth variety X is Frobenius split, it suffices to show that there is a section $\varphi \in H^0(X, \omega_X^{1-p})$ which locally "looks like" $t^{\mathbf{p-1}}$. This is equivalent to finding a section of $\varphi \in H^0(X, \omega_X^{-1})$ which locally "looks like" $t_1 \cdots t_n$. Such a section will have "order of vanishing" equal to 1 in the t_i direction for each i. We make this precise with the following proposition. First we introduce some terminology.

Definition 4.10. Let X be a smooth variety of dimension n and let $Y_1, \dots, Y_r$ be prime divisors of X (closed subvarieties of codimension 1) with ideal sheaves $\mathcal{I}_{Y_1}, \dots, \mathcal{I}_{Y_r}$. We say that $Y_1, \dots, Y_r$ *intersect transversely* at some $x \in X$ if there exists a system of local coordinates $t_1, \dots, t_n$ for x such that in some neighborhood $U \subset X$ containing x, we have that $\mathcal{I}_{Y_i}|_U \cong (\tilde{t_i})$, so Y_i is locally defined by the equation $t_i = 0$.

Proposition 4.11. *Let X be a proper smooth variety of dimension n. Suppose there exists $\sigma \in H^0(X, \omega_X^{-1})$ with divisor of zeros and poles*

$$\operatorname{div}(\sigma) = Y_1 + \cdots + Y_n + Z \tag{4.41}$$

where $Y_1, \ldots, Y_n$ are prime divisors intersecting transversely at some $x \in X$ and Z is an effective divisor not containing x. Then $\sigma^{p-1} \in H^0(X, \omega_X^{1-p})$ splits X, and $Y_1, \ldots, Y_n$ and simultaneously compatibly split.

Proof. Choose a system of local coordinates for $x \in X$ such that there exists an open neighborhood $U \subset X$ of x such that the ideal sheaf $\mathcal{I}_{Y_i}$ satisfies $\mathcal{I}_{Y_i}|_U \cong \tilde{(t_i)}$. Then as Z is effective and does not contain x, we have that

$$\sigma_x = t_1 \cdots t_n g(t_1, \ldots, t_n)\mathbf{dt}^{-1} \tag{4.42}$$

where $g(t_1, \ldots, t_n) \in k[[t_1, \ldots, t_n]]$ is such that $g(0, \ldots, 0) \neq 0$ (as otherwise Z would not be an effective divisor not containing x). Normalize so that $g(0, \ldots, 0) = 1$. Thus the coefficient of $t^{\mathbf{p-1}}$ in σ_x^{p-1} is 1, so σ^{p-1} is a splitting of X by Lemma 4.9.

Now, let $\varphi = \hat{\imath}(\sigma^{p-1})$ be the Frobenius splitting associated with σ^{p-1}. Then by Lemma 4.7, we have that

$$\begin{aligned} \varphi(t_i f(t_1, \ldots, t_n)) &= \operatorname{Tr}(t_i f(t_1, \ldots, t_n) t^{\mathbf{p-1}} g(t_1, \ldots, t_n)^{p-1}) \\ &= t_i \operatorname{Tr}(t_i^{1-p} t^{\mathbf{p-1}} f(t_1, \ldots, t_n) g(t_1, \ldots, t_n)^{p-1}). \end{aligned} \tag{4.43}$$

Since $\mathcal{I}_{Y_i}|_U \cong \tilde{(t_i)}$, we have that $\varphi(F_* \mathcal{I}_{Y_i}|_U) \subseteq F_* \mathcal{I}_{Y_i}|_U$ so $Y_i \cap U$ is compatibly split by $\varphi|_U$, so by Lemma 3.5, Y_i is compatibly split by φ. □

5 Frobenius splitting of Schubert varieties

We combine the results from the previous sections to prove our main theorem.

Proposition 5.1. *There exists a section $\sigma \in H^0(Z(w), \omega_{Z(w)}^{-1})$ such that σ^{p-1} splits $Z(w)$, and compatibly splits $Z(w(J))$ for all J.*

Proof. Let $\sigma_1 \in H^0(Z(w), \mathcal{O}_{Z(w)}(\partial Z(w)))$ be the canonical section, so that

$$\operatorname{div}(\sigma_1) = \partial Z(w). \tag{5.1}$$

Since ρ is dominant, we have that $H^0(G/B, \mathcal{L}(\rho)) \neq 0$, so there exists a section $\sigma_2 \in H^0(G/B, \mathcal{L}(\rho))$ such that $(\sigma_2)_{1 \cdot B} \neq 0$ as G/B is homogeneous. We then have that $\theta_w^* \sigma_2 \in H^0(Z(w), \mathcal{L}_w(\rho))$, and $\operatorname{div}(\theta_w^* \sigma_2) = Z$ for some effective divisor Z not containing $Z(\hat{j})$ for all j. Thus by Proposition 2.28 we have that

$$\sigma = \sigma_1 \otimes \theta_w^* \sigma_2 \in H^0\left(Z(w), \omega_{Z(w)}^{-1}\right). \tag{5.2}$$

and

$$\operatorname{div} \sigma = \partial Z(w) + Z = Z(w(\hat{1})) + \cdots + Z(w(\hat{r})) + Z. \tag{5.3}$$

By (2.30) it follows that $Z(w(\hat{1})), \ldots, Z(w(\hat{r}))$ intersect transversely at $[1, \ldots, 1]$. As Z is an effective divisor not containing $[1, \ldots, 1]$, we may apply Proposition 4.11, so we have that $Z(w)$ compatibly splits $Z(w(\hat{\imath}))$ for all i. Now applying Lemma 3.7 and using (2.30) gives the result.

□

The Frobenius splitting for Schubert varieties now follows readily from the splitting of the BSDH varieties.

Proof of Theorem 1.1. We first prove the theorem for $P = B$. Let $w_0 \in W$ be the longest element. We want to apply Lemma 3.6 using the map $\theta_{w_0} : Z(w_0) \to G/B$, since we know $Z(w_0)$ is Frobenius split by the previous proposition. Since θ_{w_0} is a birational projective morphism and G/B is normal (it is smooth), we have that $(\theta_{w_0})_* \mathcal{O}_{Z(w_0)} = \mathcal{O}_{G/B}$ by [Har77, Corollary III.11.4]. Let $w_0 = s_{i_1} \cdots s_{i_N}$ be a reduced expression for w_0. For all $w \in W$, we have that $w = s_{i_{j_1}} \cdots s_{i_{j_m}}$ is a reduced expression for w for some subsequence $j_1 < \cdots < j_m$. Setting $J = \{j_1, \ldots, j_m\}$,we have that $Z(w_0(J)) \subset Z(w_0)$ and

$$\theta_{w_0}(Z(w_0(J))) = S(w). \tag{5.4}$$

Thus by Proposition 5.1 and Lemma 3.6, $S(w)$ is Frobenius split.

Now, for an arbitrary parabolic subgroup P, recall that the natural projection map $\pi_P : G/B \to G/P$ is a locally trivial P/B-fibration and P/B is a projective variety (so its global sections are constant), so

$$(\pi_P)_* \mathcal{O}_{G/B} = \mathcal{O}_{G/P}. \tag{5.5}$$

Given any Schubert variety $S^P(w)$ in G/P, we have that $\pi_P(S(w)) = S^P(w)$, and we know that $S(w)$ is Frobenius split, so applying Lemma 3.6 gives the result. □

6 Consequences of splitting

We need the following result, which allows us to translate our results from characteristic p to characteristic 0.

Lemma 6.1. *Let G/k be a semisimple algebraic group over an algebraically closed field k of characteristic 0, P a parabolic subgroup, $S^P(w) \subset G/P$ a Schubert variety, and $\mathcal{L}$ a line bundle on $S^P(w)$. Then*

(i) There exists a scheme X which is flat over $\operatorname{Spec} \mathbb{Z}$, and a line bundle $\underline{\mathcal{L}} \in \operatorname{Pic}(X)$ such that $X_k \cong S^P(w)$, and $\underline{\mathcal{L}}_k = \mathcal{L}$, where the subscript denotes the base change to k.

(ii) There exists an open subset $U \subset \operatorname{Spec} \mathbb{Z}$ such that for all closed points $p \in U$, the geometric fiber of X over p (the base change to $\overline{\mathbb{F}_p}$) is a Schubert variety over $\overline{\mathbb{F}_p}$, and $\underline{\mathcal{L}}_{\overline{\mathbb{F}_p}}$ is a line bundle on $X_{\overline{\mathbb{F}_p}}$.

Proof. (i): This is a standard application of the theory of Chevalley group schemes. See [MR85, §3, Lemma 3] and [Ses83, Theorem 2] for more details.

(ii): This follows from the fact that for any prime p, the reduced induced structure on the geometric fiber $(X_{\overline{\mathbb{F}_p}})_{\mathrm{red}}$ is a Schubert variety over $\overline{\mathbb{F}_p}$. But being integral is an open condition (see [Gro66, Theorem 12.2.1 (x)]) and $X_{\overline{\mathbb{Q}}}$ is a Schubert variety, hence integral, so there exists an open set $U \subset \operatorname{Spec} \mathbb{Z}$ such that for every closed $p \in U$, $(X_{\overline{\mathbb{F}_p}})_{\mathrm{red}} = X_{\overline{\mathbb{F}_p}}$ is a Schubert variety. □

Using this Lemma and the result in characteristic p (Proposition 3.9), Theorem 1.2 follows from a straightforward semicontinuity argument.

Proof of Theorem 1.2. Parts (i) and (ii) for k of characteristic $p > 0$ follow from Theorem 1.1 and Proposition 3.9, so assume that $\operatorname{char} k = 0$.

(i): Assume the same setup as Lemma 6.1 and let $i \geq 1$. Since field extensions are flat, by flat base change [Har77, Theorem III.12.11] we have that

$$H^i(X_{\mathbb{Q}}, \mathcal{L}_{\mathbb{Q}}) \otimes_{\mathbb{Q}} k \cong H^i(S^P(w), \mathcal{L}) \tag{6.1}$$

Then by Lemma 6.1, there exists some prime p for which $X_{\overline{\mathbb{F}_p}}$ is a Schubert variety over $\overline{\mathbb{F}_p}$, so $H^i(X_{\overline{\mathbb{F}_p}}, \mathcal{L}_{\overline{\mathbb{F}_p}}) = 0$ by Theorem 1.1. By flat base change, we have that

$$H^i(X_{\overline{\mathbb{F}_p}}, \mathcal{L}_{\overline{\mathbb{F}_p}}) = H^i(X_{\mathbb{F}_p}, \mathcal{L}_{\mathbb{F}_p}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p} = 0 \tag{6.2}$$

so $H^i(X_{\mathbb{F}_p}, \mathcal{L}_{\mathbb{F}_p}) = 0$. By semicontinuity [Har77, Theorem III.12.8] we have that $H^i(X_{\mathbb{Q}}, \mathcal{L}_{\mathbb{Q}}) = 0$, so $H^i(S^P(w), \mathcal{L}) = 0$.

(ii): Let $Y \subset X$ be a closed subvariety with sheaf of ideals $\mathcal{I}_Y$ and let $\mathcal{L}$ be a line bundle on X such that $H^1(X, \mathcal{L}) = 0$. By the long exact sequence of cohomology applied to the exact sequence

$$0 \longrightarrow \mathcal{I}_Y \otimes \mathcal{L} \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}|_Y \longrightarrow 0$$

we have that $H^0(X, \mathcal{L}) \to H^0(Y, \mathcal{L})$ is surjective if and only if $H^1(X, \mathcal{I}_Y \otimes \mathcal{L}) = 0$. Since $\mathcal{I}_Y \otimes \mathcal{L}$ is coherent, we can apply the same semicontinuity argument as in part (i) to get the result for Schubert varieties in characteristic 0.

□

6.1 Schubert varieties are normal

In this section we prove Theorem 1.3. We first prove a relative criterion for the normality of a Frobenius split scheme.

Lemma 6.2. *Let $f : Z \to X$ be a proper morphism of varieties over an algebraically closed field k of characteristic $p > 0$. If Z is normal, the fibers of f are connected, and X is Frobenius split, then X is normal.*

Proof. Let $\nu : \tilde{X} \to X$ be the normalization of X. Then $f : Z \to X$ factors through ν. Since f has connected fibers, so does ν. But ν is a finite morphism as it is a normalization, so the fibers are discrete, and hence ν is a bijection on the underlying topological spaces.

In order to show that ν is an isomorphism, it suffices to check the claim locally, so we may assume that $X = \operatorname{Spec} A$, $\tilde{X} = \operatorname{Spec} B$. Then B is the integral closure of A in its field of fractions $K = \operatorname{Frac} A$, so we have that $A \subset B \subset K$. Now, since X is Frobenius split, the splitting φ is a map $\varphi : A \to A$ satisfying $\varphi(a_1 a_2^p) = a_2 \varphi(a_1)$ and $\varphi(1) = 1$ for all $a_1, a_2 \in A$. The key idea is that we can uniquely extend φ to a map $\varphi : K \to K$ satisfying $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(xy^p) = y\varphi(x)$ for all $x, y \in K$ by setting $\varphi(r/s) = s^{-1}\varphi(rs^{p-1})$ for $r/s \in K$ with $r, s \in A$. In particular, φ is a splitting of $\operatorname{Spec} B$ and of the localizations of A and B at all prime ideals. Now, let

$$I := \{a \in A \mid aB \subset A\} \tag{6.3}$$

be the *conductor* of B/A. I is an ideal of B contained in A (so it is also an ideal of A).

Suppose for the sake of contradiction that $B \neq A$, so that $I \neq A$ and $\operatorname{Spec} A/I$ and $\operatorname{Spec} B/I$ are closed subschemes of $\operatorname{Spec} A$ and $\operatorname{Spec} B$. Then f restricts to a bijective map $\operatorname{Spec} B/I \to \operatorname{Spec} A/I$. However, f is not an isomorphism on any of the stalks by the definition of I.

We have that $I \subset \varphi(I)$ as if $i \in I$, then $\varphi(i^p) = i$. Also, if $i \in I$ and $b \in B$, then $\varphi(i)b = \varphi(ib^p) \in \varphi(A) = A$ as $\varphi(A) = A$ by definition, so $\varphi(I) \subset I$. Thus $\varphi(I) = I$, so φ is a splitting of A

and B compatible with A/I and B/I, respectively, so the closed subschemes $\operatorname{Spec} A/I$ and $\operatorname{Spec} B/I$ are reduced by Lemma 3.2. Let $\mathfrak{p} \in \operatorname{Spec} A/I$ be a minimal prime ideal of A containing I. Then since A/I and B/I are reduced, $(A/I)_{\mathfrak{p}}$ is a field, and $(B/I)_{\mathfrak{p}}$ is a field extension. This extension is nontrivial and purely inseparable (see [Sta25, Lemma 0BRA]). Thus there exists $x \in B_{\mathfrak{p}}$ such that $\overline{x}^p \in (A/I)_{\mathfrak{p}}$ but $\overline{x} \notin (B/I)_{\mathfrak{p}}$, where $\overline{x}$ is the image of x in $(B/I)_{\mathfrak{p}} \cong B_{\mathfrak{p}}/I_{\mathfrak{p}}$. So then $x^p \in A_{\mathfrak{p}}$, but then $x = \varphi(x^p) \in \varphi(A_{\mathfrak{p}}) = A_{\mathfrak{p}}$. This is a contradiction as then $\overline{x} \in (B/I)_{\mathfrak{p}}$. So we must have that $B = A$, so $X = \tilde{X}$ is normal.

□

The normality of Schubert varieties now follows from a simple inductive argument which crucially uses Lemma 2.26.

Proof of Theorem 1.3. First, let k be of characteristic 0 and assume the same setup as Lemma 6.1. Now, the geometric fibers of X being normal is an open condition (see [Gro66, Theorem 12.2.4 iv]). Thus if $X_{\overline{\mathbb{F}_p}}$ is normal for some prime p then $X_{\overline{\mathbb{Q}}} = S^P(w)$ is normal. This shows that it suffices to prove the theorem in characteristic $p > 0$.

So let k have characteristic $p > 0$. First, the morphism $\pi_P : G/B \to G/P$ is a locally trivial fibration with smooth fiber P/B, so $\pi_P^{-1}(S^P(w))$ is normal if and only if $S^P(w)$ is normal. But $\pi_P^{-1}(S^P(w))$ is a Schubert variety, so it suffices to prove the result for the Schubert varieties $S(w)$ in G/B.

We proceed by induction on $\ell(w) = n$. If $n = 1$, then $S(w) \cong \mathbb{P}^1$ is smooth. So suppose that $\ell(w) = n > 1$ and the claim holds for all Schubert varieties $S(w')$ with $\ell(w') < n$. Let $s_i \in W$ be a simple reflection such that $\ell(ws_i) = \ell(w) - 1$. By Lemma 2.26, $S(w)$ is a locally trivial $\mathbb{P}^1$-fibration over $S^{P(s_i)}(w)$. Additionally, $\pi_{P(s_i)}(S(ws_i)) = S^{P(s_i)}(w)$, and the map $\pi_{P(s_i)}|_{S(ws_i)} : S(ws_i) \to S^{P(s_i)}(w)$ is birational with connected fibers.

By induction, $S(ws_i)$ is normal. $S^{P(s_i)}(w)$ is Frobenius split by Theorem 1.1, so X is normal by Lemma 6.2. Then since $S(w)$ is a locally trivial $\mathbb{P}^1$-fibration over $S^{P(s_i)}(w)$, it is also normal. □

6.2 Schubert varieties are Cohen-Macaulay

In this section we prove Theorem 1.4. To do so, we define the notion of a *rational resolution*, and show that the existence of a rational resolution implies Cohen-Macaulayness in Lemma 6.6. We then prove Theorem 1.4 assuming the key Proposition 6.7, which we prove in Section 6.3. The material in this section is adapted from [Ram85] and [BK04, §3.2-3.4]. Throughout this section we utilize the higher direct image sheaf and the Leray spectral sequence. See [Har77, §III.8] for information on the higher direct image sheaf and [Sta25, Section 01EY] for information on the Leray spectral sequence.

We first define the notion of a *rational morphism*, which is referred to as a *trivial morphism* in [Ram85] and [Kem76] (our notation follows [BK04]).

Definition 6.3. Let $f : X \to Y$ be a morphism of schemes. Then f is a *rational morphism* if $\mathcal{O}_Y = f_*\mathcal{O}_X$ and $R^i f_*(\mathcal{O}_X) = 0$ for all $i \geq 1$.

The composition of two rational morphisms is rational, and the base change of a rational morphism by a flat morphism is a rational morphism (see [Kem76, §3]).

A rational resolution is now just a resolution of singularities which is a rational morphism, and such that the higher direct images of ω_X vanish.

Definition 6.4. Let $f : X \to Y$ be a proper, birational, rational morphism of varieties. If X is smooth and for all $i \geq 1$,

$$R^i f_*(\omega_X) = 0, \tag{6.4}$$

then f is a *rational resolution*.

The vanishing of higher direct images interacts nicely with cohomology.

Lemma 6.5. *Let $f : X \to Y$ be a morphism of schemes, $\mathcal{F}$ an $\mathcal{O}_X$-module and $\mathcal{G}$ a locally free sheaf on Y. Then*

(i) If $R^i f_*(\mathcal{F}) = 0$ for all $i \geq 1$, then $H^p(X, \mathcal{F}) \cong H^p(Y, f_*\mathcal{F})$ for all $p \geq 0$.

(ii) If $R^i f_*(\mathcal{O}_X) = 0$ for all $i \geq 1$, then $H^i(X, f^*\mathcal{G}) \cong H^i(Y, \mathcal{G})$.

Proof. (i): This follows easily from the Leray spectral sequence for $f : X \to Y$ (almost all the terms in the spectral sequence are 0).

(ii): By the projection formula [Har77, Exercise III.8.3], we have that

$$R^i f_*(f^*\mathcal{G}) \cong R^i f_*(\mathcal{O}_X \otimes f^*\mathcal{G}) \cong R^i f_*(\mathcal{O}_X) \otimes \mathcal{G} = 0. \tag{6.5}$$

By part (i) and the projection formula again, we have that

$$H^i(X, f^*\mathcal{G}) \cong H^i(Y, f_*f^*\mathcal{G}) \cong H^i(Y, \mathcal{G}) \tag{6.6}$$

as desired. □

Using this lemma and Serre duality, we are able to relate rational resolutions to Cohen-Macaulayness.

Lemma 6.6. *Let $f : X \to Y$ be a rational resolution. Then Y is Cohen-Macaulay.*

Proof. By [Har77, Theorem III.7.6], Y it suffices to show that for any ample line bundle $\mathcal{L}$ on Y we have that $H^i(Y, \mathcal{L}^{-n}) = 0$ for $i < \dim Y$ and n sufficiently large. By Lemma 6.5 (i), we have that $H^i(Y, \mathcal{L}^{-n}) \cong H^i(X, f^*\mathcal{L}^{-n})$. By Serre duality [Har77, Theorem III.7.6], we have that $H^i(X, f^*\mathcal{L}^{-n})$ is dual to $H^{\dim X - i}(X, \omega_X \otimes f^*\mathcal{L}^n)$, so it suffices to show that the latter vanishes. Now, by the projection formula and the fact that $R^i f_*(\omega_X) = 0$ because f is a rational resolution, for $i \geq 1$ we have that

$$R^i f_*(\omega_X \otimes f^*\mathcal{L}^n) \cong R^i f_*\omega_X \otimes \mathcal{L}^n = 0. \tag{6.7}$$

Applying Lemma 6.5 (i) and the projection formula again gives

$$H^j(X, \omega_X \otimes f^*\mathcal{L}^n) \cong H^j(Y, f_*\omega_X \otimes f_*f^*\mathcal{L}^n) \cong H^j(Y, f_*\omega_X \otimes f_*\mathcal{O}_X \otimes \mathcal{L}^n) \tag{6.8}$$

for all $j \geq 0$. As $\mathcal{L}$ is ample, by Serre's vanishing theorem this is 0 for $j \geq 1$ and n sufficiently large. □

Thus we have essentially reduced the proof of Theorem 1.4 to showing that the BSDH resolutions are rational resolutions.

Proposition 6.7. *Let k have characteristic p. Then the BSDH resolution $\theta_w : Z(w) \to S(w)$ is a rational resolution.*

The proof of Theorem 1.4 follows easily from this proposition, as we now detail. We prove Proposition 6.7 in the following section, as it requires a significant amount of extra work. We extensively use the Frobenius splitting of Schubert varieties in the proof. In particular, we utilize Lemma 3.2, Lemma 3.7, and Proposition 3.9.

Proof of Theorem 1.4. By [Gro66, Theorem 12.2.1 (vii)], Cohen-Macaulayness is an open condition, so arguing as in the proof of Theorem 1.3, we may assume that k has characteristic $p > 0$.

Again arguing as in the proof of Theorem 1.3, the morphism $\pi_P : G/B \to G/P$ is a locally trivial fibration with smooth fiber P/B, so $\pi_P^{-1}(S^P(w))$ is Cohen-Macaulay if and only if $S^P(w)$ is Cohen-Macaulay. But $\pi_P^{-1}(S^P(w))$ is a Schubert variety, so it suffices to prove the result for the Schubert varieties $S(w)$ in G/B. Then the result follows immediately from Lemma 6.6 and Proposition 6.7. □

6.3 Proof of Proposition 6.7

In this section, we prove Proposition 6.7, thereby completing the proof of Theorem 1.4. First we give two rationality criteria.

Lemma 6.8. *Let $f : X \to Y$ be a morphism of projective varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Y$, $Z \subset X$ a closed subvariety, and $\mathcal{L}$ an ample line bundle on Y. Let $W = f(Z)$ be the scheme theoretic image of Z under f, and let $g = f|_Z$ be the restriction of f to Z. Suppose for all sufficiently large n and all $p \geq 1$, $H^p(Z, g^*\mathcal{L}^n) = 0$. Then g is a rational morphism.*

Proof. We use the Leray spectral sequence applied to the morphism f and the sheaf $\mathcal{O}_Z \otimes \mathcal{L}^n$. We have that the (p, q) term of the second page of the Leray spectral sequence is

$$E_2^{p,q} \cong H^p(Y, R^q f_*(\mathcal{O}_Z \otimes \mathcal{L}^n)) \cong H^p(Y, (R^q f_*\mathcal{O}_Z) \otimes \mathcal{L}^n) \tag{6.9}$$

where the second equivalence is by the projection formula. Thus by Serre's vanishing theorem, if n is sufficiently large we have that $E_2^{p,q} = 0$ for all $p > 0$. Then by the Leray spectral sequence we have that

$$H^q(X, \mathcal{O}_Z \otimes f^*\mathcal{L}^n) \cong H^q(Z, g^*\mathcal{L}^n) \cong H^0(Y, (R^q g_*\mathcal{O}_Z) \otimes \mathcal{L}^n). \tag{6.10}$$

By assumption, this is 0 for all $q > 0$ and all n sufficiently large. But since $\mathcal{L}$ is ample, for n sufficiently large $\mathcal{L}^n$ is generated by global sections, so we must have that $R^q g_*\mathcal{O}_Z = 0$ for all $q > 0$. ☐

Lemma 6.9. *Let $f : X \to Y$ be a proper morphism of schemes, and let $Z \subset X$ be a closed subscheme with ideal sheaf $\mathcal{I}_Z$. Suppose f is a rational morphism. The following are equivalent:*

(i) For all $i \geq 1$, $R^i f_*\mathcal{I}_Z = 0$.

(ii) $f|_Z : Z \to f(Z)$ is a rational morphism.

If either of these conditions holds, then $f_*\mathcal{I}_Z$ is the ideal sheaf of $f(Z)$.

Proof. Applying the higher direct image long exact sequence gives

$$0 \longrightarrow f_*\mathcal{I}_Z \longrightarrow f_*\mathcal{O}_X \longrightarrow f_*\mathcal{O}_Z \longrightarrow R^1 f_*\mathcal{I}_Z \longrightarrow R^1 f_*\mathcal{O}_X \longrightarrow \cdots$$

As f is a rational morphism we have that $R^i f_*\mathcal{O}_X = 0$ for all $i \geq 1$, so $R^{i-1} f_*\mathcal{O}_Z \cong R^i f_*\mathcal{I}_Z$ for all $i \geq 1$, which gives the claim. ☐

We are now ready to complete the proof.

Proof of Proposition 6.7. Let $w = s_{i_1} \cdots s_{i_n}$ be a reduced expression for $w \in W$, so that $v = s_{i_2} \cdots s_{i_n}$ is a reduced expression for $v \in V$. Let $f : Z(w) \to Z(v)$ be the projection $[p_1, \ldots, p_n] \mapsto [p_2, \ldots, p_n]$ and let $\sigma = i_{w(1)} : Z(v) \to Z(w)$ mapping $[p_2, \ldots, p_n] \mapsto [1, p_2, \ldots, p_n]$ be a section for f. Let $\pi : G/B \to B/P(s_{i_1})$ be the projection. We first show the following claim.
Claim. θ_w and $\theta_w|_{f^{-1}(\partial Z(v))}$ are rational morphisms.
Proof. We proceed by induction on $n = \ell(w)$. The base case $n = 1$ is trivial as θ_w is an isomorphism. Assume the claim holds up to $n - 1$. We have the pullback diagram

$$\begin{array}{ccc} Z(w) & \xrightarrow{\theta_w} & S(w) \\ \sigma \uparrow \; \downarrow f & & \downarrow \pi \\ Z(v) & \xrightarrow{\pi \circ \theta_v} & \pi(S(w)) \end{array}$$

Thus θ_w and $\theta_w|_{f^{-1}(\partial Z(v))}$ are the respective base changes of $\pi|_{S(v)} \circ \theta_v$ and $\pi|_{\partial S(v)} \circ \theta_v$ by π. Since π is flat, rationality is preserved under flat base change, θ_v is rational by the inductive hypothesis, and the composition of two rational morphisms is rational, it suffices to show that $\pi|_{S(v)}$ and $\pi|_{\partial S(v)}$ are rational morphisms.

Let $\mathcal{L}$ be an ample line bundle on $G/P(s_{i_1})$. Then $\pi^*\mathcal{L}$ is an ample line bundle on G/B, as is the restriction of $\pi^*\mathcal{L}$ to $S(v)$ and $\partial S(v)$. Then since Schubert varieties are Frobenius split, the unions of Schubert varieties are Frobenius split by Lemma 3.7. Then by Proposition 3.9, $\pi^*\mathcal{L}$ has vanishing higher cohomology when restricted to $S(v)$ and $\partial S(v)$ as $\partial S(v)$ is a union of Schubert varieties. Thus we can apply Lemma 6.8, so $\pi|_{S(v)}$ and $\pi|_{\partial S(v)}$ are rational morphisms, which completes the proof of the claim. □

By the claim, to complete the proof of the proposition we just need to show that $R^i(\theta_w)_*(\omega_X) = 0$ for $i \geq 1$. By Proposition 2.28, we have that

$$\omega_{Z(w)} \cong \mathcal{O}_{Z(w)}(-\partial Z(w)) \otimes \theta_w^*\mathcal{L}(-\rho). \tag{6.11}$$

Applying the projection formula gives

$$R^i(\theta_w)_*\omega_{Z(w)} \cong R^i(\theta_w)_*\mathcal{O}_{Z(w)}(-\partial Z(w)) \otimes \mathcal{L}(-\rho) \tag{6.12}$$

so it suffices to show that $R^i(\theta_w)_*\mathcal{O}_{Z(w)}(-\partial Z(w)) = 0$ for all $i \geq 1$. Now, $\mathcal{O}_{Z(w)}(-\partial Z(w))$ is the ideal sheaf of $\partial Z(w)$ in $Z(w)$. Thus by Lemma 6.9 it suffices to show that $(\theta_w)|_{\partial Z(w)}$ is a rational morphism. We proceed by induction on $n = \ell(w)$. The base case $n = 1$ is trivial as $\partial Z(w)$ is just a point.

So assume the claim holds up to $n - 1$. We have the closed immersion exact sequence

$$0 \longrightarrow \mathcal{O}_{Z(w)}(-\sigma(Z(v))) \longrightarrow \mathcal{O}_{Z(w)} \longrightarrow \sigma_*\mathcal{O}_{Z(v)} \longrightarrow 0$$

Taking the tensor product by $f^*\mathcal{O}_{Z(v)}(-\partial Z(v))$ gives (recall that σ is a section of f)

$$0 \longrightarrow \mathcal{O}_{Z(w)}(-\partial Z(w)) \longrightarrow f^*\mathcal{O}_{Z(v)}(-\partial Z(v)) \longrightarrow \sigma_*\mathcal{O}_{Z(v)}(-\partial Z(v)) \longrightarrow 0$$

By the claim and Lemma 6.9, the middle term has vanishing higher direct images. As $\theta_v = \theta_w \circ \sigma$, the last term has higher vanishing direct images by the inductive hypothesis. Thus taking the long exact sequence of higher direct images we find that $R^i(\theta_w)_*\mathcal{O}_{Z(w)}(-\partial Z(w)) = 0$ for $i \geq 2$.

It remains to show that $i = 1$ case. It suffices to show that

$$(\theta_w)_*f^*\mathcal{O}_{Z(v)}(-\partial Z(v)) \to (\theta_w)_*\sigma_*\mathcal{O}_{Z(v)}(-\partial Z(v)) \tag{6.13}$$

is surjective. Since θ_w is a rational morphism, by Lemma 6.9 we have that $(\theta_w)_*f^*\mathcal{O}_{Z(v)}(-\partial Z(v))$ is the ideal sheaf of $X := \theta_w f^{-1}(\partial Z(v)) = \bigcup_{r \neq 1} S(w(\hat{r}))$ in $S(w)$. Likewise, $Y := (\theta_w)_*\sigma_*\mathcal{O}_{Z(v)}(-\partial Z(v))$ is the ideal sheaf of $\partial S(v)$ in $S(v)$ as $\theta_w \circ \sigma = \theta_v$. As X and Y are both the unions of Schubert varieties, by Lemma 3.7 they are Frobenius split, so by Lemma 3.2 they are reduced. Since $X \cap S(v) = Y$ is also reduced by Lemmas 3.7 and 3.2, it follows that (6.13) is surjective, as it is the restriction of the surjective map $\mathcal{O}_{S(w)} \to \mathcal{O}_{S(v)}$ to the ideal sheaf $(\theta_w)_*f^*\mathcal{O}_{Z(v)}(-\partial Z(v)) = \mathcal{I}_X$, and the image will be the ideal sheaf $(\theta_w)_*\sigma_*\mathcal{O}_{Z(v)}(-\partial Z(v)) = \mathcal{I}_Y$. □

7 Conclusion

In this essay we proved several nice geometric properties about Schubert varieties using their Frobenius splitting, but we have not exhausted all applications. For instance, we can introduce the notion of splitting relative to a divisor, which is essentially a Frobenius splitting φ which factors through a section of $F_*\mathcal{O}_X(D)$, where D is an effective Cartier divisor [RR85]. With a bit more work, we can show that the splittings of Schubert varieties given in Theorem 1.1 are splitting relative to a divisor, which implies the vanishing of cohomology for all *semi-ample* line bundles (a line bundle $\mathcal{L}$ is semi-ample if $\mathcal{L}^n$ is generated by global sections for some n). While we showed that they are Cohen-Macaulay, the Frobenius splitting of Schubert varieties also implies that they are *arithmetically Cohen-Macaulay* [Ram85]. There is also the notion of a *diagonal splitting* of a variety X, which is a splitting of $X \times X$ which is compatible with the diagonal [Ram87]. The flag varieties G/P are diagonally split, but it is an open question if the Schubert varieties $S^P(w)$ are as well. The existence of a diagonal splitting of X implies that all ample line bundles on X are very ample. For information on all these applications and more, see [BK04].

The Frobenius splitting of other types of schemes is also a topic of study, and one can derive similar geometric consequences as a result. For instance, Brion and Inamdar [BI94] study the Frobenius splitting of spherical varieties, which are generalizations of Schubert varieties. Brion and Kumar's book [BK04] contains many other examples, such as the Frobenius splitting of the Hilbert scheme of a nonsingular Frobenius split surface.

The geometry of Schubert varieties has applications in other areas as well. In representation theory, the Weyl character formula gives an elegant formula for the dimensions of the weight spaces of an irreducible representation $V(\lambda)$ of a semisimple group G. A Demazure module is a submodule of $V_w(\lambda)$ whose construction can be seen as an analogue of a Schubert variety $S(w) \subset G/B$. The Demazure character formula gives a description of the character of a Demazure module analogous to the Weyl character formula. Demazure's original proof [Dem74] was found to have serious gaps, and Anderson [And85] proved the formula using the geometric properties of Schubert varieties developed in this paper (in particular, as a consequence of their Frobenius splitting).

As a final application, Schubert varieties play an important role in the study of Shimura varieties. Shimura varieties can be thought of as generalizations of modular curves, and are widely studied for their role in the Langlands program and other areas of number theory. In many cases, the special fiber of these local models is isomorphic to a union of affine Schubert varieties [PZ13]. Affine Schubert varieties are generalizations of the classical Schubert varieties studied in this essay, which arise from the affine Grassmannian. By studying the tangent spaces of these of these Schubert varieties, one can classify the reduction type of Shimura varieties at a prime. For more information, see [PRS13].

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