Abelian Varieties

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These notes are based on a course of the same title given by Professor Tony Scholl at Cambridge during Lent Term 2025. They have been written up by Alexander Shashkov. I have added details to certain proofs which we did not cover in full, and made a few off-hand remarks. As a result, there are likely plenty of errors, which are my own.

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1 Introduction

1.1 Motivation

Let E be an elliptic curve over a field k. This means that E is a proper, irreducible curve over k whose points form a group. We have the following facts:

(i) E can be embedded as a nonsingular cubic $E \hookrightarrow \mathbb{P}^2_k$.

(ii) The group law on E is commutative.

(iii) If $k = \overline{k}$ is algebraically closed, $E(k)$ is a divisible group and $\forall n \geq 1$, $E[n] = \{x \in E(k) | nx = 0\}$ is finite. If $\text{char}(k) = 0$ or $\text{char}(k) = p$ and $p \nmid n$, $E[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$. If $\text{char}(k) = p$, then $E[p^r] \cong \mathbb{Z}/p^r\mathbb{Z}$ or 0.

  Definition 1.1. A divisible group G is a group where any element is divisible by any positive integer. So for all $x \in G$ and $n \in \mathbb{Z}_{\geq 1}$, there exists $y \in G$ such that $ny = x$.

(iv) $E(k) \cong \text{Cl}^0(E)$ with the isomorphism given by $P \to (P) - (0)$. This is known as Abel's theorem, or the Abel-Jacobi theorem.

  Definition 1.2. The degree 0 divisor class group $\text{Cl}^0(E)$ is the group of all degree 0 divisors divided by the group of all degree 0 principal divisors. In many cases (and for all projective varieties) it seems like all principal divisors have degree 0.

Most of the above facts can be proved using the elementary methods, namely the equation defining an elliptic curve and the group law, as well as the Riemann-Roch theorem.

An Abelian Variety is a higher dimensional analogue of an elliptic curve. This means that it is a proper variety X over a field k whose points form a group (in some sense). We have the following analogues of the 4 properties of elliptic curves listed above:

1. X is a projective variety, but there might not be a nice set of equations defining it.
2. The group law (which we have not defined) is commutative.
3. If $k = \overline{k}$ is algebraically closed, then $X(k)$ is a divisible group and $X[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2 \dim X}$ if $\text{char}(k) \nmid n$, and if $p = \text{char}(k) > 0$, then $X[p^r] \cong (\mathbb{Z}/p^r\mathbb{Z})^j$ for some $0 < j \leq \dim X$.
4. There exists another abelian variety $\widehat{X}$, called the *dual abelian variety*, such that $\widehat{X}(k) \cong \text{Pic}^0(X) \cong \text{Cl}^0(X) \subset \text{Pic}(X)$. Additionally, there is a surjective homomorphism $X \to \widehat{X}$ with finite kernel.

The first half of this course is a continuation of algebraic geometry. We will study two different topics:

1. We will prove more things about the cohomology of coherent sheaves.
2. We will find out that the Riemann-Roch theorem is about two different things:
   (a) $\dim H^0(X, \mathcal{L}) - \dim H^1(X, \mathcal{L}) = 1 - g + \deg \mathcal{L}$.
   (b) $H^1(X, \mathcal{L})^\vee \cong H^0(X, \Omega^1_X \otimes \mathcal{L})$.

1.2 Conventions

- Rings are commutative with 1, and ring homomorphisms take $1 \to 1$.
- If $f : A \to B$ is a ring homomorphism, then B is an A-algebra.
- An A-module is *finite* if it is finitely generated, so there exists a surjective homomorphism $A^n \to M$.
- An A-algebra B is *finite* if it is finitely generated as an A-module.
- An A-algebra B is *finite type* if it is finitely generated over A, so there exists a surjective homomorphism $\varphi : A[x_1, \dots, x_n] \to B$.
- A finite type A-algebra B has finite presentation if the homomorphism φ defined above is finitely generated.
- If A is Noetherian, finite type implies finite presentation.
- For most of this course, we can assume that rings are Noetherian.
- A family of elements of a set S is $(x_i)_{i \in I}$, or a function $I \to S$, for some set I.

2 Varieties over a field

In classical algebraic geometry, a variety is a closed irreducible subset of $\mathbb{A}^n(k) \cong k^n$ or $\mathbb{P}^n(k)$, and we take k to be algebraically closed.

- We also might take an open subset of a projective variety, which is a quasi-projective variety.
- There exist proper schemes which are non projective.
- Even in classical AG, non-algebraically closed fields appear. Take some morphism $f : X \to S$. We consider X as a parameter space, and the fibers of f (meaning the schemes $X_s = X \times_Y \operatorname{Spec} k(s)$ for each s) can be not algebraically closed. I think he is talking about the residue fields $k(s)$ specifically?

For the purposes of this course, we use the following definition of a k-variety.

Definition 2.1. Let k be a field, and $\overline{k}$ its algebraic closure. A k-variety (or a variety over k) is a k-scheme X which is separated, of finite type, and such that $X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ is integral (reduced and irreducible).

By separated and of finite type we mean that the structure morphism $X \to k$ is separated and of finite type.

The condition that $X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ is called being *geometrically integral.* In general, being geometrically [property] means that the base change to $\overline{k}$ has [property].

Example 2.2. If $k = \mathbb{R}$, then $X = \operatorname{Spec} \mathbb{R}[T_1, T_2]/(T_1^2 + T_2^2)$ is not a variety. The base change to $\mathbb{C}$ is reducible.

Example 2.3. If $k = \mathbb{F}_p(t)$ is a function field, then $X = \operatorname{Spec} k[T_1, T_2]/(T_1^p - tT_2^p)$ is not a variety. $X_{\overline{k}}$ is not reduced because $T_1^p - tT_2^p = (T_1 - t^{1/p}T_2)^p$ in $\overline{k}$.

Proposition 2.4. *If X and Y are k-varieties, then so is $X \times_k Y$.*

Proof. First we reduce to the case where $X = \operatorname{Spec} A, Y = \operatorname{Spec} B$ (exercise).

In this case we have that $X \times_k Y = \operatorname{Spec} A \otimes_k B$, this is of finite type because A and B are of finite type (just take the tensor product of the generators of A and B).

We have that $X \times_k Y$ is separated because it is affine (Hartshorne Prop 2.4.1).

It remains to show that $\operatorname{Spec} A \otimes_k B$ is geometrically integral. It suffices to show that

$$(A \otimes_k B) \otimes_k \overline{k} = (A \otimes_k \overline{k}) \otimes_{\overline{k}} (B \otimes_k \overline{k}) \tag{2.1}$$

is an integral domain. By assumption $A \otimes_k \overline{k}$ and $B \otimes_k \overline{k}$ are integral domains. So it suffices to show that if $k = \overline{k}$ is algebraically closed, and A, B are finite type k-algebras and integral domains, then $A \otimes_k B$ is an integral domain.

If not, then there exists some relation

$$(\sum a_i \otimes b_i)(\sum a'_j \otimes b'_j) = 0. \tag{2.2}$$

WLOG, we may assume that the sets $\{b_i\}$ and $\{b'_i\}$ are each linearly independent, and a_1, a'_1 are nonzero. Thus $a_1 a'_1 \neq 0$, so the distinguished open set $D(a_1 a'_1) \neq 0$ in $\operatorname{Spec} A$. Thus there exists a maximal ideal $\mathfrak{m} \in D(a_1 a'_1)$, so $a_1 a'_1 \notin \mathfrak{m}$, and $A/\mathfrak{m} = k$ because k is algebraically closed. Here we use that there exists a maximal ideal not containing $a_1 a'_1$, this is true because A is a finite type k-algebra which is a domain, so the intersection of all the maximal ideals is $\sqrt{(0)} = (0)$. Also, we use that $A/\mathfrak{m}$ is an algebraic extension of k, which is a consequence of the Nullstellensatz.

Anyways, we reduce our zero relation mod $\mathfrak{m}$, so we have that

$$(\sum \overline{a_i} \otimes b_i)(\sum \overline{a'_j} \otimes b'_j) = 0 \tag{2.3}$$

in $(A/\mathfrak{m}) \otimes_k B \cong k \otimes_k B = B$. But since B is a domain, this means that $\sum \overline{a_i} \otimes b_i = 0$. But then by the linear independence of the b_is, we have that $\overline{a_i} = 0$ for all i, so in particular $\overline{a_1} = 0$, so $a_1 \in \mathfrak{m}$, so $a_1 a'_1 \in \mathfrak{m}$, so $a_1 a'_1 \in \mathfrak{m}$, which is a contradiction. □

Let X be a k-variety, so it is an integral scheme which has a unique generic point $\eta \in X$, and $\mathcal{O}_{X,\eta} = k(X)$ is a field, the function field of X. Now, given a cover of X by affine schemes $U_i = \operatorname{Spec} A_i$, we have that A_i is a finitely generated k-algebra, and $k(X)$ is the fraction field of A_i, so $k(X)/k$ is finitely generated over k as a field extension (this means there is a surjective morphism $k(T_1, \ldots, T_n) \to k(X)$.

Proposition 2.5. *If X is a k-variety, then k is algebraically closed in $k(X)$.*

Proof. This means that for any $a \in k(X)$, a being algebraic over k implies that $a \in k$.

Suppose not. Then there exists $a \in k(X)$ which is algebraic over k, such that $k \subsetneq k(a) \subset k(X)$. Since $k(X) = \mathcal{O}_{X,\eta}$, there exists a nonempty open affine $\operatorname{Spec} A = U \subset X$ such that $a \in A \subset k(X)$. Explicitly, we can take any open affine $V = \operatorname{Spec} B$, and a will be an element of $\operatorname{Frac}(B)$, so $a = b/b'$ with $b, b' \in B$, and then we can take $U = D(b') = \operatorname{Spec} B_{b'}$.

We then have that $\operatorname{Spec}(A \otimes_k \overline{k}) = U_{\overline{k}} \subset X_{\overline{k}}$, and since $k(a) \subset A$ because A is a k-algebra, we have that $k(a) \otimes_k \overline{k} \subset A \otimes_k \overline{k}$. But since $a \notin k$ and $\overline{k}$ is the algebraic closure, $a \in \overline{k}$ and $k(a) \otimes_k \overline{k}$ is not an integral domain, so $A \otimes_k \overline{k}$ is not an integral domain, so $X_{\overline{k}}$ is not an integral scheme, contradicting geometric integrality. □

Remark 2.6. The converse theorem is not true in general, but is true if k is perfect.

Over a non-perfect field, there can exist X integral, separated, finite type over k, but not geometrically integral, so not a variety, such that k is algebraically closed in $k(X)$.

Definition 2.7.
1. A variety X is *projective* if there exists a closed immersion $X \hookrightarrow \mathbb{P}^n_k$.
2. A variety X is *affine* if there exists a closed immersion $X \hookrightarrow \mathbb{A}^n_k$.
3. A variety is quasi projective if it is an open subscheme of a projective variety.

Theorem 2.8. *$\mathbb{P}^n_k$ is a proper scheme over $\operatorname{Spec} k$. More generally, $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is proper.*

Proof. Hartshorne Theorem 2.4.9. □

Corollary 2.9. *Every projective k-variety X is proper over k.*

Proof. By Hartshorne Corollary 2.4.8, a closed immersion if proper, and the composition of two proper morphisms is proper, so $X \hookleftarrow \mathbb{P}^n_k \to k$ is proper. □

We next state a "basic finiteness theorem" which will be useful for the next result.

Theorem 2.10. *Let $f : X \to \operatorname{Spec} A$ be a proper morphism, A Noetherian, and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then $\Gamma(X, \mathcal{F})$ is a* finite *A-module.*

I'm not sure how hard it is to prove this. We need it for the next result.

Theorem 2.11. *If X is a proper k-variety, then $\Gamma(X, \mathcal{O}_X) = k$.*

Proof. Applying Theorem 2.10, we have that $\Gamma(X, \mathcal{O}_X) = B$ is a finite k-algebra, and $B \subset k(X)$ because X is integral. But since $k \subset B \subset k(X)$ and B is a finite k-algebra, B is a finite field extension of k contained in $k(X)$, so by Proposition 2.5, $B = k$. □

There are many proper varieties which are not projective. Hironaka gave a few examples which can be found in Hartshorne.

Next we have two useful results.

Definition 2.12. A morphism $f : X \to Y$ of integral schemes is *birational* if there exists a nonempty open set $V \subset Y$ such that $f^{-1}(V) \to V$ is an isomorphism.

Note that $f^{-1}(V) \subset X$ and $V \subset Y$ are dense open sets because X, Y are integral and thus irreducible topological spaces.

Lemma 2.13 (Chow)**.** *If X is a proper k-variety, then there exists a birational morphism $f : X' \to X$ where X' is a projective k-variety.*

We will prove this below.

Theorem 2.14 (Nagata)**.** *Every k-variety is an open subscheme of a proper k-variety.*

This theorem is much harder to prove. Here is also a more general version of Chow's lemma, a proof of which can be found in the Stacks project.

Lemma 2.15 (General Chow's)**.** *Let S be a Noetherian scheme, $g : X \to S$ a proper morphism. Then $\exists$ a surjective $f : X' \to X$ such that X' is a closed subscheme of $\mathbb{P}^n_S$ and there exists an open dense $U \subseteq X$ such that $f : f^{-1}(U) \to U$ is an isomorphism.*

Now we sketch a proof of Chow's lemma.

Lemma 2.13. Let $X = \bigcup_{i=1}^m U_i$ be an affine open cover. We can take a finite open cover because X is of finite type over k. Since each $U_i = \operatorname{Spec} A_i$ is a finitely generated k-algebra, for each i, $U_i \hookrightarrow \mathbb{A}_k^{n_i}$ for some n_i, as $A_i = k[x_1, \ldots, x_{n_i}]/I_i$ for some ideal I_i. Let $U_i^* \subseteq \mathbb{P}_k^{n_i}$ be the closure of U_i in $\mathbb{P}_k^{n_i}$, with the reduced subscheme structure. This is the unique structure on U_i^* which makes $U_i^* \to \mathbb{P}_k^{n_i}$ a closed immersion, and such that U_i^* is a reduced scheme. The construction is essentially by constructing a quasi-coherent sheaf of ideals, and then taking U_i^* to be the associated closed subscheme.

Then $U_i^* \cap \mathbb{A}_k^{n_i} = U_i$ and U_i^* is a variety (why?). So now let $U = \bigcap U_i \hookrightarrow \prod \mathbb{A}_k^{n_i}$ be the "diagonal embedding". This is a closed immersion.

I'm going to leave the rest of this proof for later because I don't really understand it. □

3 Differentials and nonsingularity and smoothness

Definition 3.1. Let $\varphi : A \to B$ be a ring map. Then there exists a B-module $\Omega_{B/A}$, the module of (Kähler) differentials, together with a derivation $d = d_{B/A} : B \to \Omega_{B/A}$, which means an A-linear map satisfying $d(xy) = ydx + xdy$. $\Omega_{B/A}$ is universal for all A-derivations of B, which means that if M is a B-module, and $\operatorname{Der}_A(B, M)$ is the set of all A-derivations $D : B \to M$, then

$$\operatorname{Hom}_B(\Omega_{B/A}, M) \cong \operatorname{Der}_A(B, M) \tag{3.1}$$

via $\psi \to D = \psi \circ d_{B/A}$. Another way of saying this is that $\Omega_{B/A}$ is initial in the slice category, or satisfies the obvious universal property.

We construct $\Omega_{B/A}$ in two different ways:

1. Generators and relations: We set $\Omega_{B/A} = P/Q$ where P is the free B-module on $\{[b] | b \in B\}$ and Q is the submodule generated by $[\varphi(a)]$ for all $a \in A$ and $[b_1 b_2] - b_1[b_2] - b_2[b_1]$ and $[b_1 + b_2] - [b_1] - [b_2]$. Then $d_{B/A}(b)$ is the image of $[b]$ in P/Q.

2. Let $\mu : B \otimes_A B \to B$ be the standard multiplication map $b \otimes b' \to bb'$, $J = \ker \mu$. There are two maps $B \to B \otimes_A B$ whose composition with μ is the identity, them being $b \to 1 \otimes b$ and $b \to b \otimes 1$. Then J/J^2 is a $B \otimes_A B$-module annihilated by J, so the two B-module structures on it are the same. In particular, we have that if $a \otimes c \in J$, then $(1 \otimes b - b \otimes 1)(a \otimes c) \in J^2$, so defining $b(a \otimes c) = a \otimes bc$ or $b(a \otimes c) = ab \otimes c$ makes no difference. We then define our derivation

$$\begin{aligned} d' : B &\to J/J^2 \\ b &\to 1 \otimes b - b \otimes 1. \end{aligned} \tag{3.2}$$

Proposition 3.2. *There exists an isomorphism $P/Q \cong J/J^2$, $[b] \mod Q \to d'(b)$.*

Remark 3.3. Let $C = B \otimes_A B/J^2$. Then $B \to B \otimes_A A \hookrightarrow C$, I don't understand this remark.

Remark 3.4. The differential has nice functorial properties. If

$$\begin{array}{ccc} B & \longrightarrow & B' \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array}$$

is a commutative square of ring maps, there is an induced B-module map $\Omega_{B/A} \to \Omega_{B'/A'}$ which is transitive for $B \to B' \to B''$ and induces a B'-module map by extension of scalars: $\Omega_{B/A} \otimes_B B' \to \Omega_{B'/A'}$.

Proposition 3.5. *If $B' = B \otimes_A A'$, then $\Omega_{B/A} \otimes_A A' \cong \Omega_{B/A} \otimes_B B' \cong \Omega_{B'/A'}$.*

Proof. Exercise. □

In particular, if $S \subset A$ is a multiplicatively closed set, and S_B is the image of S in B, then

$$\Omega_{S_B^{-1}B/S^{-1}A} = S_B^{-1} \cdot \Omega_{B/A} \tag{3.3}$$

Example 3.6. If $B = A[t_1, \dots, t_n]$ is a polynomial algebra, then

$$\Omega_{B/A} = \bigoplus B(dt_i) \tag{3.4}$$

is a free module on the symbols dt_i. Since $B \otimes_A B = A[\{t_i \otimes 1, 1 \otimes t_i\}] = A[\{t_i \otimes 1, z_i\}]$ where $z_i = 1 \otimes t_i - t_i \otimes 1$, we have that J is the ideal generated by $\{z_i\}$.

Next we define two exact sequences.

Proposition 3.7. *Let $A \to B \to C$ be a ring map. Then*

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0 \tag{3.5}$$

is an exact sequence of C-modules, where the map $\Omega_{C/A} \to \Omega_{C/B}$ is given by $d_{C/A}c \to d_{C/B}c$.

Proposition 3.8. *If $A \to B \to C = B/I$ be ring maps, where $B \to C$ is given by the obvious $b \to b + I$, then*

$$I/I^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0 \tag{3.6}$$

is an exact sequence of C-modules, where the first map takes $f + I^2 \to df \otimes 1$.

The proof of both of these propositions is left as an exercise.

Corollary 3.9. *If $B = A[\{x_i\}]$, $C = B/(\{f_j\})$, then*

$$\Omega_{C/A} = \bigoplus C dx_i / (\sum_j df_j) \tag{3.7}$$

where $df_j = \sum_i \frac{\partial f_j}{\partial x_i} dx_i$.

Example 3.10 (Differentials and separability)**.** Let L/K be a finite extension of fields. Then L/K is separable if and only if $\Omega_{L/K} = 0$.

We write $K \subset K_1 \subset L$ such that $K \subset K_1$ is separable and $K_1 \subset L$ is purely inseparable. Then $K_1 = K(\alpha) = K[t]/(g)$ with $g(\alpha) = 0$, $g'(\alpha) \neq 0$, g irreducible.

Applying Corollary 3.9, we have that

$$\Omega_{K_1/K} = K_1 dt/K_1 g'(t)dt = (K(\alpha)d\alpha)/(K(\alpha)g'(\alpha)d\alpha) = 0 \tag{3.8}$$

because $g'(\alpha) \neq 0$. Then by Proposition 3.7 with ring map $K \to K_1 \to L$, we have $\Omega_{L/K_1} = \Omega_{L/K}$. Then if L/K is separable, we have that $K_1 = L$, so then $\Omega_{L/K} = 0$.

Now, if L/K is purely inseparable, then $K_1 = K$ and $K \subset K_2 \subsetneq L$, where $L = K_2(\beta) = K_2[t]/(h)$ with $h(t) = t^p - b$, $p = \text{char}(K) \neq 0$. Then $\Omega_{L/K_2} = Ld\beta/(Lh'(\beta)d\beta = Ld\beta$ because $h'(\beta) = 0$, so $\Omega_{L/K} \neq 0$ by Proposition 3.7.

Proposition 3.11. *If $A \to B$ is a ring map, $S \subset B$ multiplicatively closed, then*

$$S^{-1}B \otimes_B \Omega_{B/A} = S^{-1}\Omega_{B/A} \cong \Omega_{S^{-1}B/A} \tag{3.9}$$

3.1 Sheafification

We now construct the Kahler differential for a general scheme S. Let $f : X \to Y$ be a morphism of schemes. We will define a quasi-coherent sheaf $\Omega_{X/Y}$ of $\mathcal{O}_X$-modules and a $f^{-1}\mathcal{O}_Y$-linear map

$$d = d_{X/Y} : \mathcal{O}_X \to \Omega_{X/Y}, \tag{3.10}$$

the sheaf of relative differentials, or the relative cotangent sheaf. This sheaf will be functorial for commutative squares, so if

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

is a commutative square and g, g' are open immersions, then $(g')^*\Omega_{X/Y} \to \Omega_{X'/Y'}$ is an isomorphism.

Also if $X = \operatorname{Spec} B \to \operatorname{Spec} A = Y$ is a morphism of affine schemes, then $\Omega_{X/Y} = \widetilde{\Omega_{B/A}}$.

Definition 3.12. Recall that if $X = \operatorname{Spec} R$, M an R-module, then $\tilde{M}$ is a sheaf of $\mathcal{O}_X$-modules such that $\Gamma(X, \tilde{M}) = M$, $\Gamma(D(f), \tilde{M}) = M_f$, and $(\tilde{M})_{\mathfrak{p}} = M_{\mathfrak{p}}$.

If $f : \operatorname{Spec} R' \to \operatorname{Spec} R$, then $f^*\tilde{M} = \widetilde{M \otimes_R R'}$. In particular, if $U = \operatorname{Spec} R' \subset \operatorname{Spec} R$ is an open affine, then $\tilde{M}(U) = M \otimes_R R'$.

A quasi-coherent sheaf is one which locally looks like $\tilde{M}$.

Now we begin constructing our differential. First assume that f is separated. Then the diagonal map $\Delta : X \to X \times_Y X = X^2$ is a closed immersion, so we have an ideal sheaf $\mathcal{I}_\Delta \subseteq \mathcal{O}_{X^2}$, such that $\mathcal{O}_{X^2}/\mathcal{I}_\Delta = \Delta_*\mathcal{O}_X$. Now, by some result on the Stacks project (29.31/01R1), since $\mathcal{I}_\Delta/\mathcal{I}_\Delta^2$ is annihilated by $\mathcal{I}_\Delta$, we have that $\mathcal{I}_\Delta/\mathcal{I}_\Delta^2 = \Delta_*\Omega_{X/Y}$ for a unique quasi-coherent $\mathcal{O}_X$-module $\Omega_{X/Y}$.

Now, suppose that $j : X' \to X$, $Y' \to Y$ are open immersions such that $f(X') \subset Y'$. We can check that $\Omega_{X'/Y'} = \Omega_{X/Y}|_{X'} = j^*\Omega_{X/Y}$. In fact, if $\Delta' : X' \to X' \times_{Y'} X'$ then $\mathcal{I}_{\Delta'} = \mathcal{I}_\Delta|_{X' \times X'}$, so $\mathcal{I}_{\Delta'}/\mathcal{I}_{\Delta'}^2 = (\mathcal{I}_\Delta/\mathcal{I}_\Delta^2)|_{X' \times X'}$. Thus $\Omega_{X'/Y'} = \Omega_{X/Y}|_{X'}$.

If $X = \operatorname{Spec} B \to Y = \operatorname{Spec} A$, then $\mathcal{I}_\Delta = \tilde{J}$, where $J = \ker(\mu : B \otimes_A B \to B)$, so $\Omega_{X/Y} = \widetilde{J/J^2} = \widetilde{\Omega_{B/A}}$.

Now for the general case where f might not be separable. First we sketch the following nice lemma about Δ.

Definition 3.13. Let $f : X \to Y$ be a morphism of schemes. Then f is an immersion if it is a closed immersion followed by an open immersion.

Remark 3.14. Not every immersion can be written in the reversed form as an open immersion followed by a closed immersion. But it is hard to find an example of this happening.

Lemma 3.15. *Let $f : X \to Y$ be a morphism of schemes. The diagonal map $\Delta : X \to X \times_Y X$ is an immersion of schemes.*

Proof. Write $X = \bigcup U_i$, $U_i = \operatorname{Spec} A_i$ such that $f(U_i) \subset V_i$ for some open affine V_i of Y. Then we have

$$i : X \to V = \bigcup U_i \times_{V_i} U_i. \tag{3.11}$$

But $i^{-1}(U_i \times_Y U_i) = U_i$, so i is a closed immersion because it is basically the union of the diagonal embedding of affine schemes, which is a closed immersion (see the stacks). Further, we have that

$$j : \bigcup U_i \times_{V_i} U_i \to X \times_Y X \tag{3.12}$$

is an open immersion by some other result in the stacks, so $\Delta = j \circ i$ is an immersion. □

Now consider $\mathcal{I}_{X \subset V}$ where V is a defined above to be the quasi-coherent ideal sheaf of $\mathcal{O}_V$-modules, and define $\Omega_{X/Y}$ to be the associated quasi-coherent $\mathcal{O}_X$-module $i^{-1}(\mathcal{I}_{X\subset V}/\mathcal{I}_{X\subset V}^2)$. This is independent of V because it vanishes away from the diagonal. Also, the functoriality of the diagonal implies the functoriality properties.

To define the differential $d = d_{X/Y} : \mathcal{O}_X \to \Omega_{X/Y}$, it is enough to define it on each open affine $U = \operatorname{Spec} B \subset X$ such that $f(U) \subset V = \operatorname{Spec} A \subset Y$ some open affine. We define $\mathcal{O}_X(U) = B \to \Omega_{X/Y}(U) = \Omega_{B/A}$ by just taking $d_{B/A}$. Then since we have a local morphism of sheaves, we can extend it to a global one since it is nice.

Now that we have our construction finished, we can prove Proposition 3.11.

Proposition 3.11. Let $X = \operatorname{Spec} B$ and $X' = \operatorname{Spec} S^{-1}B$ be schemes over $Y = \operatorname{Spec} A$.

If S is generated by a finite subset, then X' is open in X. Then $\Omega_{X/Y} = \widetilde{\Omega_{B/A}}$ so $\Omega_{X'/Y} = \Omega_{X/Y}|_{X'} = \widetilde{S^{-1}\Omega_{B/A}}$ by the properties of the twiddle operator. So $\Omega_{S^{-1}B/A} = \Omega_{X'/Y}(X') = S^{-1}\Omega_{B/A}$.

For an arbitrary, potentially infinite S, we pass to the limit. □

Since $\Omega_{X/Y} = \widetilde{\Omega_{B/A}}$ if $X = \operatorname{Spec} B \to Y = \operatorname{Spec} A$, we can rewrite the two exact sequences in terms of sheaves.

Proposition 3.16. *Let*

$$X \xrightarrow{f} Y \longrightarrow S$$

be a morphism of schemes. Then we have an exact sequence

$$f^*\Omega_{Y/S} \longrightarrow \Omega_{X/S} \longrightarrow \Omega_{X/Y} \longrightarrow 0.$$

Proposition 3.17. *Let $i : Z \to X$ be a closed subscheme given by a quasi-coherent sheaf of ideals $\mathcal{I} = \mathcal{I}_{Z/X} = \mathcal{O}_X$, $f : X \to Y$ a morphism of schemes. Then we have an exact sequence*

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow i^*\Omega_{X/Y} \longrightarrow \Omega_{Z/Y} \longrightarrow 0.$$

Next we show that $\Omega_{X/Y}$ is coherent in certain nice situations.

Definition 3.18. A scheme X is locally Noetherian if every $x \in X$ has an affine open neighborhood $\operatorname{Spec} R$ with R Noetherian. An equivalent condition is that every affine open is Noetherian.

Let X be a locally Noetherian scheme. Then a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is coherent if for every open affine $\operatorname{Spec} R = U \subset X$, we have that $\mathcal{F}|_U = \tilde{M}$ for $M = \mathcal{F}(U)$ a finite (finitely generated) R-module.

It is enough to check coherence for a single open affine cover of X.

Remark 3.19. Hartshorne has the wrong definition of coherence for schemes which are not Noetherian.

Proposition 3.20. *Let $f : X \to Y$ be a morphism of schemes which is locally of finite type, and Y be locally Noetherian. Then $\Omega_{X/Y}$ is coherent.*

Proof. First note the conditions of the proposition imply that X is locally Noetherian, since for any $\operatorname{Spec} A = V \subset Y$ we have that $f^{-1}(V)$ can be covered by finitely generated A-algebras, which will have the form $B = A[T_1, \dots, T_n]/I$, and by Hilbert's basis theorem B is Noetherian. Thus we can cover X by Noetherian open affines.

The question is local on X, so we can assume that $X = \operatorname{Spec} B \to Y = \operatorname{Spec} A$ with A, B Noetherian, and in particular $B = A[T_1, \dots, T_n]/I$. It suffices to show that $\Omega_{B/A}$ is a finite B-module since the differential is functorial with respect to open immersions.

By Proposition 3.8, we have surjection

$$\Omega_{A[T_1,\dots,T_n]/A} \otimes_A B = \bigoplus_{i=1}^{n} B dT_i \to \Omega_{B/A}. \tag{3.13}$$

Since $\bigoplus B dT_i$ is a finite B-module (it's a free module of rank n) we have that $\Omega_{B/A}$ is a finite B-module. □

3.2 Tangent and cotangent spaces

Let X be a k-scheme, so we have a structure morphism $X \to \operatorname{Spec} k$, and let $x \in X$ be any point. Then we have a residue field $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ which will be a field extension of k since morphisms of schemes correspond to morphisms of stalks. We will write x for the morphism $\operatorname{Spec} k(x) \to X$ given by mapping $* \to x$, induced by the map of stalks $\mathcal{O}_{X,x} \to k(x) = \mathcal{O}_{X,x}/\mathfrak{m}$.

If $x \in X$ is a closed point this is a morphism of k-schemes.

Now, if $\mathcal{F}$ is any $\mathcal{O}_X$-module, we define

$$\mathcal{F}(x) = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x) = \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x. \tag{3.14}$$

Definition 3.21. The *cotangent space* to X at $x \in X$ is the $k(x)$ vector space

$$T^*_{X/k,x} = \Omega_{X/k}(x). \tag{3.15}$$

Note that this is not the dual to the tangent space T in general.

Example 3.22. Let $X = \mathbb{A}^n_k$, then $\Omega_{X/k} = \widetilde{\Omega_{k[T_1,\ldots,T_n]/k}}$ is a free $\mathcal{O}_X$-module with basis $dT_1, \ldots dT_n$. So for any point $x \in X$, we have $T^*_{\mathbb{A}^n_k/k,x} = \bigoplus k(X)dT_i$ since we have that

$$\begin{aligned}\Omega_{X/k}(x) &= \left(\bigoplus_{i=1}^n \mathcal{O}_X dT_i\right)_x \otimes_{\mathcal{O}_{X,x}} k(x) \\ &= \left(\bigoplus_{i=1}^n \mathcal{O}_{X,x} dT_i\right) \otimes_{\mathcal{O}_{X,x}} k(x) \\ &= \bigoplus_{i=1}^n k(x) dT_i. \end{aligned} \tag{3.16}$$

In scheme theory, nilpotents play the role that infinitesimals play in calculus.

Definition 3.23. Let K be any field. Then the ring of *dual numbers* over K is $K[\epsilon] = K[T]/(T^2) = K \oplus K\epsilon$. This is a local ring with maximal ideal (ϵ).

The scheme $\operatorname{Spec} K[\epsilon]$ is non-reduced, and consists of only 1 point (ϵ). There is a canonical closed reduced subscheme $\operatorname{Spec} K \to \operatorname{Spec} K[\epsilon]$ given by the morphism $K[\epsilon] \to K$ sending $\epsilon \to 0$.

Definition 3.24. Let X be any k-scheme. The *(Zariski) tangent space* to X at $x \in X$ is the set

$$T_{X/k,x} = \{k\text{-scheme morphisms } \theta : \operatorname{Spec} k(x)[\epsilon] \to X \text{ such that } \theta|_{\operatorname{Spec} k(x)} \text{ is } x : \operatorname{Spec} k(x) \to X\} \tag{3.17}$$

The idea here is that we start with some $x : \operatorname{Spec} k(x) \to X$ and enlarge it to a morphism $\theta : \operatorname{Spec} k(x)[\epsilon] \to X$. Intuitively, we think of the embedding of $\operatorname{Spec} k(x)[\epsilon]$ into X as a point x plus a direction.

Theorem 3.25. *There is a canonical bijection*

$$T_{X/k,x} \to \operatorname{Hom}_{k(x)}(T^*_{X/k,x}, k(x)) = \operatorname{Hom}_{\mathcal{O}_{X,x}}(\Omega_{X/k,x}, k(x)) \tag{3.18}$$

so that T is the dual of T^* (but maybe not the other way).

Proof. By localization, we have that $\Omega_{X/k,x} = \Omega_{\mathcal{O}_{X,x}/k}$ and since $\operatorname{Spec} k(x)[\epsilon]$ is one point, every θ of the form (3.17) factors through $\mathcal{O}_{X,x} \to X$. So we can replace X by $\operatorname{Spec} \mathcal{O}_{X,x}$.

Thus WLOG we have $X = \operatorname{Spec} A$, where $(A, \mathfrak{m}, k(x) = K = A/\mathfrak{m})$ is a local k-algebra, $x \in X$ is the closed point (the maximal ideal). Then

$$T_{X/k,x} = \{k\text{-algebra homomorphisms } \varphi : A \to K[\epsilon] \text{ such that } \forall a \in A,\ \varphi(a) \mod \epsilon \overline{a}\} \tag{3.19}$$

where $\overline{a}$ is given by $A \to A/\mathfrak{m} = K$, $a \to \overline{a}$. Any such φ is of the form $\varphi(a) = \overline{a} + D(a)\epsilon$, for some map $D : A \to K$ which is k-linear. Now, φ is a homomorphism if and only if

$$\begin{aligned}\varphi(aa') &= \overline{aa'} + D(aa')\epsilon \\ &= \varphi(a)\varphi(a') \\ &= \overline{aa'} + (\overline{a}D(a') + \overline{a'}D(a))\epsilon \end{aligned} \tag{3.20}$$

if and only if D is a k-derivation $A \to k$. So $\varphi \to D$ is a bijection so we have a bijection

$$T_{X/k,x} \to \operatorname{Der}_k(A, k) = \operatorname{Hom}_A(\Omega_{A/k}, k). \tag{3.21}$$

Thus in particular $T_{X/k,x}$ is a $k(x)$-vector space. □

Remark 3.26. The base field k doesn't play much of a role here. We can replace $\operatorname{Spec} k$ by any base scheme S and define $T^*_{X/S,x} = \Omega_{X/S}(x)$ for $f : X \to S$ and $T_{X/S,x}$ to be the set of morphisms θ as above.

However, this generality is illusory because of the functoriality properties of the Kahler differential (basically the local property of it). If $f(x) = s \in S$, then if $X_s = X \times_S \operatorname{Spec} k(s)$ is the fibre at s, then we have a commuting square

$$\begin{array}{ccc} X_s & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k(s) & \longrightarrow & S \end{array}$$

and we have that $T^*_{X/S,x} = T^*_{X_s/k(s),x}$ and $T_{X/S,x} = T_{X_s/k(s),x}$.

Proposition 3.27. *Let X be a k-scheme, $x \in X$ with the residue field $k(x) = k$. Then $\Omega_{X/k}(x) \cong \mathfrak{m}_x/\mathfrak{m}_x^2$ as k-vector spaces.*

Proof. Both sides depend only on $\mathcal{O}_{X,x}$. Thus we may assume that $X = \operatorname{Spec} A$, where $(A, \mathfrak{m})$ is a local k-algebra with $A/\mathfrak{m} = k$. Consider the second exact sequence Proposition 3.8 or 3.17 for $k \to A \to A/\mathfrak{m} = k$. Then we have an exact sequence

$$\mathfrak{m}/\mathfrak{m}^2 \to \Omega_{A/k} \otimes_A k = T^*_{X/k,x} \to \Omega_{k/k} = 0 \tag{3.22}$$

so we have a surjection $\mathfrak{m}/\mathfrak{m}^2 \to T^*_{X/k,x}$.

Thus it suffices to show that the map $\mathfrak{m}/\mathfrak{m}^2 \to \Omega_{A/k} \otimes_A k$ is injective, or equivalently, that the dual map

$$\operatorname{Hom}_A(\Omega_{A/k}, k) \to \operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k) \tag{3.23}$$

is surjective. Further, recall that by the universal property for $\Omega_{A/k}$, we have that $\operatorname{Hom}_A(\Omega_{A/k}, k) = \operatorname{Der}_k(A, k)$.

If $D \in \operatorname{Der}_k(A, k)$ and $x, y \in \mathfrak{m}$, then $D(xy) = (x+\mathfrak{m})D(y) + (y+\mathfrak{m})D(x) = 0$ because $x, y \in \mathfrak{m}$. Thus we have that $\operatorname{Der}_k(A, k) = \operatorname{Der}_k(A/\mathfrak{m}^2, k)$, because $D \in \operatorname{Der}_k(A, k)$ vanishes on $\mathfrak{m}^2$.

As the sequence $0 \to \mathfrak{m} \to A \to A/\mathfrak{m} = k \to 0$ is exact, it splits so $A = k \oplus \mathfrak{m}$. Further, we have that $A/\mathfrak{m}^2 = k \oplus (\mathfrak{m}/\mathfrak{m}^2)$ with multiplication given by

$$(a, x)(b, y) = (ab, ay + bx) \tag{3.24}$$

which basically follows from

$$(a + x)(b + y) = ab + (ay + bx) + xy \tag{3.25}$$

and we have that $ab \in k$, $ay + bx \in \mathfrak{m}$, and $xy \in \mathfrak{m}^2$ so it is zero.

Thus given some $\varphi \in \operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$, we can extend it to a $D \in \operatorname{Der}_k(A/\mathfrak{m}^2, k)$ by writing $D = \varphi \circ \pi$, where π is the projection map $A/\mathfrak{m}^2 = k \oplus \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2$. $\square$

Remark 3.28. If more generally $k(x)/k$ is a finite separable extension, then Proposition 3.27 still holds with the isomorphism as $k(x)$-vector spaces, since $\Omega_{k(x)/k} = 0$ by Example 3.10 and $A/\mathfrak{m}^2 \cong k(x) \oplus (\mathfrak{m}/\mathfrak{m}^2)$ from Example sheet 1.

Example 3.29. If $k = \overline{k}$, then $\forall x \in X$ closed points we have that

$$T^*_{X/k,x} = \mathfrak{m}_x/\mathfrak{m}_x^2. \tag{3.26}$$

This is the definition of the cotangent space in some books.

3.3 Algebraic interlude

Lemma 3.30. *Let A be a ring, $\varphi : A^m \to A^n$ an A-module map. Then*

$$\{\mathfrak{p} \in \operatorname{Spec} A \mid \operatorname{rank}(\varphi \otimes_A k(\mathfrak{p}) : k(\mathfrak{p})^m \to k(\mathfrak{p})^n) \leq r\} \tag{3.27}$$

is a closed subset of $\operatorname{Spec} A$.

Proof. The map $\varphi \otimes k(\mathfrak{p})$ is the induced map on $k(\mathfrak{p})$-vector spaces.

The above set is equal to

$$\mathfrak{p}| \text{ all } (r+1) \times (r+1)\text{-minors of } \varphi \otimes k(\mathfrak{p}) \text{ vanish}\} = V(I), \tag{3.28}$$

where I is the ideal generated by all $(r+1) \times (r+1)$ minors of φ. Basically, the $(r+1) \times (r+1)$ minors are the determinants of dimension $r+1$-subspaces. If they all vanish in $A_{\mathfrak{p}}$ then $\varphi \otimes k(\mathfrak{p})$ has rank at most r. So if we take the ideal I containing all of them, then they vanish in $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}$ if and only if $I \subseteq \mathfrak{p}$. ☐

Proposition 3.31. *Let X be a locally Noetherian scheme, $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then*

1. *The function $x \to d(X) = \dim_{k(x)} \mathcal{F}(x)$ is upper semi-continuous.*
2. *If X reduced, then d constant implies that $\mathcal{F}$ is locally free of dimension d. Note that if $\mathcal{F}$ is locally free, then the function d is locally constant.*

Proof. 1. We need to show that (this is the definition of upper semi-continuous)

$$Z_r = \{x \in X | d(x) \geq r\} \tag{3.29}$$

is closed for all r.

We can check this locally because being closed is local on a base, so assume that $X = \operatorname{Spec} A$ is affine, A is Noetherian, and $\mathcal{F} = \widetilde{M}$ for M a finite A-module. Let

$$A^m \xrightarrow{\varphi} A^n M \longrightarrow \quad \longrightarrow 0$$

be a presentation. Then $\forall x \in X$, the induced sequence

$$k(x)^m \xrightarrow{\varphi \otimes k(x)} k(x)^n \longrightarrow M \otimes_A k(x) \longrightarrow 0$$

is exact by the right exactness of the tensor product.

So $d(x) = n - \operatorname{rank}(\varphi \otimes k(x))$. So

$$Z_r = \{x | \operatorname{rank}(\varphi \otimes k(x)) \leq n - r, \tag{3.30}$$

which is closed by the lemma.

2. First, note that the reduced condition is essential, and we can consider counter-examples as simple as $X = \operatorname{Spec} k[\epsilon]$ and $M = k$ as a $k[\epsilon]$-module.

Now, let $d = d(x)$ be constant for all $x \in X$. Being locally free is a local condition, so again we can assume that $X = \operatorname{Spec} A, \mathcal{F} = \widetilde{M}$. Suppose there exists a surjection

$$\pi : A^d \twoheadrightarrow M. \tag{3.31}$$

By hypothesis, $\pi \otimes k(\mathfrak{p}) : k(\mathfrak{p})^d \to M \otimes k(\mathfrak{p})$ is an isomorphism for all $\mathfrak{p} \in \operatorname{Spec} A$. So we have an inclusion

$$(A/\mathfrak{p})^d \hookrightarrow M/\mathfrak{p}M. \tag{3.32}$$

Thus $\ker \pi \subset \bigcap_{\mathfrak{p}} \mathfrak{p}^d \subset A^d$, and A is reduced so $\bigcap \mathfrak{p} = 0$, so $M \cong A^d$ is free.

In general, it suffices to show that for all $\mathfrak{m} \in \operatorname{Spec} A$ maximal, there exists $D(f) = \operatorname{Spec} A_f$ containing $\mathfrak{m}$ with $\mathcal{F}|_{D(f)} = \widetilde{M_f}$ free, as then $\mathcal{F}$ will be locally free.

Let $k = A/\mathfrak{m}$, and pick a basis $M \otimes_A k = \bigoplus m_i \otimes 1$, $m_i \in M$. Put $M' = \sum Am_i \subset M$, so $M = M' + \mathfrak{m}M$. Pretty much M' is M modulo $\mathfrak{m}$, so we add in $\mathfrak{M}$ to get M. Then by Nakayama's lemma, there exists $f \in 1 + \mathfrak{m}$ such that $f(M/M') = 0$. Then

$$(m_i) : A_f^d \twoheadrightarrow M'_f = M_f \tag{3.33}$$

because $A_f^d \twoheadrightarrow M'_f$ via $e_i \to m_i/1$, and $M'_f = M_f$ by Nakayama.

So by the previous step, $\tilde{M}_f$ is free. Also since $f \in 1 + \mathfrak{m}$, we have $\mathfrak{m} \in D(f)$. □

Proposition 3.32. *Let X be an integral k-scheme of finite type, $d = \dim X$. Then $\forall x \in X$, $\dim_{k(x)} T^*_{X/k,x} = \dim T_{X/k,x} \geq d$ and we say X is* smooth over k at x *if equality holds, so the dimension of the (co)tangent space is d.*

We say X is smooth over k *if it is smooth at all $x \in X$.*

Proof. Let $\eta \in X$ be the generic point of X. Then Proposition 3.31 part 1 applied to $\Omega_{X/k}$ shows that for all $x \in X$, $\dim T^*_{X/k,x} \geq \dim T^*_{X/k,\eta} = \dim_k \Omega_{K/k}$, where $K = k(x) = k(\eta)$. By question 6 on sheet 1, this is $\geq$ the transcendence degree of K/k, which is d. □

Theorem 3.33. *Let X be an integral k-scheme of finite type, $\dim X = d$. Then*

1. *The set of smooth points of X, X^{sm}, is open in X.*

2. *X is smooth at $x \in X$ if and only if $\exists U \subseteq X$ an open neighborhood of x such that $\Omega_{X/k}|_U \cong \mathcal{O}_U^d$ is free of rank d.*

   Thus X is smooth if and only if $\Omega_{X/k}$ is locally free of rank d.

3. *Let k'/k be an algebraic extension (for instance take $k' = \overline{k}$). Take the projection morphism $p : X' = X \times_k \operatorname{Spec} k' \to X$. Assume X' is integral, and for $x' \in X'$, let $x = p(x')$. Then X' is smooth over k' at x' if and only if X is smooth over k at x.*

   Note that p is surjective by the going-up theorem, so X is smooth over k if and only if X' is smooth over k'.

Proof. 1. Applying Proposition 3.31 (i) to $\Omega_{X/k}$, we find that the set of points where $\dim T^*_{X/k,x} \geq d+1$ is closed, and since $\dim T^*_{X/k,x} \geq d$ always, the set of points where $\dim T^*_{X/k,x} = d$ is open.

2. If $\Omega_{X/k}|_U \cong \mathcal{O}_U^d$, and $x \in U$, then taking stalks gives the desired result.

Now let $x \in V = X^{\mathrm{sm}}$. Then

$$\dim_{k(y)} \Omega_{X/k}(y) = d \tag{3.34}$$

for all $y \in V$, and since X is reduced, by Proposition 3.31 (ii) we have that $\Omega_{X/k}|_V$ is locally free of rank d.

3. By Proposition 3.5, we have that $\Omega_{X'/k'} = p^*\Omega_{X/k}$, so

$$T^*_{X'/k',x'} = T^*_{X/k,x} \otimes_{k(x)} k(x') \tag{3.35}$$

so the dimensions over the corresponding fields are equal and the fact that $\dim X = \dim X'$ gives the result. □

For varieties, we have a stronger result.

Theorem 3.34. *Let X be a k-variety. Then*

1. *X^{sm} is non empty, so it is a dense open subset.*
2. *$x \in X^{\mathrm{sm}}$ if and only if $\Omega_{X/k}$ is free in a neighborhood of x (and thus it will be free of dimension d in this neighborhood).*

Proof. 1. By Theorem 3.33 (iii), it is sufficient to prove that $(X \times_k \operatorname{Spec} \overline{k})^{\mathrm{sm}} \neq \emptyset$, so assume that $k = \overline{k}$.

Let $\eta \in X$ be the generic point, and let $K = k(\eta) = k(x) = \mathcal{O}_{X,\eta}$. By commutative algebra, because $k = \overline{k}$ and K/k is finitely generated, there exists $K_0 = k(t_1, \ldots, t_d) \subset X$ purely transcendental, where $d = \dim X = \mathrm{trdegK}$, with K/K_0 finite, separable. Then $\Omega_{K/k} = \Omega_{K_0/k} \otimes_{K_0} K = \bigoplus K dt_i$ by Sheet 1, question 6, and this equals $T^*_{X/k,\eta}$. Thus $\eta \in X^{\mathrm{sm}}$.

2. By the first part, X is smooth over k at η, and $\dim T^*_{X/k,\eta} = d = \dim X$.

Now, if $x \in X^{\mathrm{sm}}$ we are done by Theorem 3.33 (ii).

If $x \in U$ is open such that $\Omega_{X/k}|_U$ is free, then $\eta \in U$, and $\Omega_{X/k}|_U$ has to have rank d. So $x \in X^{\mathrm{sm}}$. □

Remark 3.35. Morally, x is a smooth point if locally, X looks like $\mathbb{A}^d_k$ in a neighborhood of x.

This is a "manifold like" condition.

Example 3.36. Consider the cubic $\operatorname{Spec} k[u, v]/(v^2 - u(u-1)^2)$, $\operatorname{char} k \neq 2$. At the point $x = (v, u-1)$, we have that $T^*_{X/k,x} \cong k^2$, but $\Omega_{X/k}$ is locally free of rank 1 on the open set $X \backslash \{x\}$, so x is the only non-smooth point.

Example 3.37. $\Omega_{X/k}$ being locally free isn't enough for smoothness in general (but it is for varieties).

Let $k = \mathbb{F}_p(t)$, $X = \operatorname{Spec} K$, where $K = k(\sqrt[p]{t}) = k(s)$, for $s^p = t$. Then X is integral, of finite type over k, and $\dim X = 0$. But we have that

$$\Omega_{X/k} = \widetilde{\Omega_{K/k}} \neq 0 \tag{3.36}$$

as $\Omega_{K/k} = K ds \neq 0$. So $\Omega_{X/k}$ is free of rank 1, but $\dim X = 0$, so X is not smooth over k.

This is because X is not geometrically integral.

Over algebraically closed $k = \overline{k}$, there is another equivalent notion of smoothness. First we recall two definitions of regularity.

Definition 3.38. Let $(A, \mathfrak{m})$ be a local ring of (Krull) dimension d with $A/\mathfrak{m} = k$. Then the following are equivalent, and if either holds we say that A is *regular*:

1. $\mathfrak{m}$ can be generated by d elements $e_1, \ldots, e_d$.

2. $\dim_k \mathfrak{m}/\mathfrak{m}^2 = d$.

In general, we have that the minimal number of generators of $\mathfrak{m}$ is at least d by Krull's principal ideal theorem, so a local ring is regular if the number of generators is minimal.

The proof of the equivalence of these things is actually contained in the proof of the next theorem.

Theorem 3.39. *Let X be an integral scheme of finite type over algebraically closed $k = \overline{k}$, and $\dim X = d$. Note that if we also assume X is separated, then X is a variety. Then $x \in X^{\mathrm{sm}}$ if and only if the local ring $\mathcal{O}_{X,x}$ is regular.*

Remark 3.40. Note that $\dim \mathcal{O}_{X,x} = \dim X$ as $x \in X$ is closed.

Also note that $\dim X = \operatorname{trdeg} k(x)/k$.

Proof. As $k(x) = k$, we have that $T^*_{X/k,x} = \mathfrak{m}_x/\mathfrak{m}_x^2$ by Proposition 3.27. Let $a_1, \dots, a_r \in \mathfrak{m}_x$. By Nakayama, $a_1, \dots, a_r$ generate $\mathfrak{m}_x$ if and only if they generate $\mathfrak{m}_x/\mathfrak{m}_x^2$. So $\mathcal{O}_X$ is regular if and only if $\dim \mathfrak{m}_x/\mathfrak{m}_x^2 = d$. □

Remark 3.41 (Warning!)**.** If k is not algebraically closed then the previous result can be false, so we can have a non-smooth point with a regular local ring.

For instance, let $X = \operatorname{Spec} K$, and let K/k be a nontrivial separable extension.

Example 3.42. The previous theorem can even be false for a k-variety if k is not algebraically closed.

Take $k = \mathbb{F}_p(a)$, $p \neq 2$, $X = \operatorname{Spec} k[u,v]/(v^2 - u^p - a) = \operatorname{Spec} A$. Then X is geometrically integral because the defining polynomial is irreducible over $\overline{k}$, and after checking the other conditions we can see that X is a k-variety of dimension 1. Let $x = \mathfrak{p} = (v, u^p - a) = (v) \subset A$. Then $A/\mathfrak{p} = k[u]/(u^p - a) = k(\sqrt[p]{a})$. So x is a closed point and $\mathfrak{m}_x = v\mathcal{O}_{X,x} \subset \mathcal{O}_{X,x}$. We then have that

$$\Omega_{A/k} = (Adu \oplus Adv)/(Avdv) = Adu \oplus (A/vA)dv \tag{3.37}$$

so $\dim_{X/k,y} = 1$ for all $y \neq x$. But $\dim T_{X/k,x} = 2$, so x is not a smooth point.

Over $\overline{k}$, the equation for X becomes $v^2 = (u - a^{1/p})^p$, which clearly isn't smooth.

Remark 3.43. Let X be an integral k-scheme of finite type over k, $x \in X$, $d = \dim X$. Then X is smooth over k at $x \in X$ if and only if there exists an open affine neighborhood of x $U = \operatorname{Spec} A \subseteq X$, with

$$A = k[t_1, \dots, t_{m+d}]/(g_1, \dots, g_m) \tag{3.38}$$

such that the matrix $M = \left(\frac{\partial g_i}{\partial t_j}(x)\right)_{ij}$ has rank m.

Given such a presentation, then $T^*_{X/k,x}$ is the cokernel of M, and should have dimension d. The other direction is harder, but standard.

3.4 Digression on finiteness and related conditions

Let X be a scheme. Recall that

- X is *quasicompact (qc)* if and only if X is a finite union of open affines. This is not really a topological condition, but a finiteness condition.

- Some non-quasicompact schemes are $\bigsqcup_{n\in\mathbb{Z}} \operatorname{Spec} k$, and $\mathbb{A}_k^\infty \backslash (0,0,\ldots)$. Note that this is the inverse limit of $\mathbb{A}_k^n$.
- If X is a separated scheme, then the intersection of any two open affines is affine.
- X is *quasiseparated* if the intersection of any 2 open affines is quasicompact.
- A non-quasiseparated scheme is $\mathbb{A}_k^\infty \cup \mathbb{A}_k^\infty$ glued together on the complement of the origin. This is the infinite dimensional version of the line with doubled origin.
- Fun exercise: find a scheme $X = U \cup V$ with U, V affine such that $U \cap V = \bigsqcup_{n\in\mathbb{Z}} \operatorname{Spec} k$. We basically want an affine scheme $\operatorname{Spec} A$ which contains $\bigsqcup_{n\in\mathbb{Z}} \operatorname{Spec} k$ as an open subset.
- Non-quasiseparated schemes rarely arise in nature, but non-separated schemes are fairly common.
- We sometimes work with schemes which are quasicompact and quasiseparated (qcqs).
- A morphism $f : X \to Y$ is quasi-compact if for all open affines $U \subset Y$, $f^{-1}(U)$ is quasi-compact.
- f is locally of finite type if for all $x \in X$, there exists open affines $x \in U = \operatorname{Spec} B \subset X$, $f(U) \subset V = \operatorname{Spec} A \subset Y$, such that B is an A-algebra of finite type.

  This is equivalent to $f^{-1}(V)$ having an open affine cover by finite type A-algebras.
- f is of finite type if it is locally of finite type and quasi-compact.
- f is locally of finite presentation if it is locally of finite type and B is a finitely presented A-algebra (if A is Noetherian this is automatic).
- f is of finite presentation if it is locally of finite presentation, quasi-compact, and quasi-separated.
- X is locally Noetherian if every open affine is Noetherian.
- X is Noetherian if it is locally Noetherian and quasi-compact.

Now let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Recall that

- $\mathcal{F}$ is quasi-coherent if $\mathcal{F}|_U \cong \tilde{M}$ where $U = \operatorname{Spec} A$ is an open affine and M is an A-module.
- $\mathcal{F}$ is of finite type if there exists a surjection $\mathcal{O}_U^n \twoheadrightarrow \mathcal{F}$.
- If X is locally Noetherian, then $\mathcal{F}$ is coherent if and only if M is finitely generated, so $\mathcal{F}$ is quasi-coherent and of finite type.
- For non-locally Noetherian schemes, there is a definition in Serre's FAC paper. The one in Hartshorne is wrong. Also, $\mathcal{O}_X$ might not be coherent.
- $\mathcal{F}$ is locally free if for each $x \in X$, there is an open neighborhood U such that $\mathcal{F}|_U$ is free.
- $\mathcal{F}$ being locally free implies that $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module for all $x \in X$.

- But the converse is false:

  Let $X = \operatorname{Spec}\mathbb{Z}$, $M = \{a/b \in \mathbb{Q} \mid b \text{ squarefree }\}$. Then $M_{(p)} = 1/p\mathbb{Z}_{(p)}$ for all primes p, so it is free. But $M \otimes_{\mathbb{Z}} \mathbb{Z}[1/n]$ is not free for any $n \geq 1$, so M is not locally free. Here, $M \otimes_{\mathbb{Z}} \mathbb{Z}[1/n]$ is the restriction of M to the affine open consisting of the primes not dividing n, all affine opens of $\operatorname{Spec}\mathbb{Z}$ are of this form.

4 Flatness and smooth morphisms

We review some things about flatness and define smooth morphisms, which are very nice.

Mumford once said "Flatness is an algebraic riddle which is the answer to many prayers".

Let A be a ring, M an A-module

Definition 4.1. M is *flat* over A (or A-flat) if for any injection $N \hookrightarrow N'$ of A-modules, $M \otimes_A N \hookrightarrow M \otimes_A N'$ is injective.

An A-algebra B is flat if it is flat as an A-module.

Here are a bunch of facts:

0. Any free A-module is flat. A flat module is one which "behaves" like a free module.

1. M is flat if and only if for every finitely generated ideal $I \subset A$, $M \otimes I \to M \otimes A = M$ is injective, so if and only if $M \otimes_A I \cong IM \subset M$. This means we can check flatness on just ideals, not all modules.

2. M if flat if $M \otimes \_\_$ is an exact functor. In general, $M \otimes \_\_$ is right exact, flatness implies left exactness.

3. Flatness is a "weak form of freeness", which can be made precise with "equational flatness":

   M is flat if and only if $\forall m_1, \ldots, m_n \in M$, $a_1, \ldots, a_n \in A$ with $\sum a_i m_i = 0$, there exists $e_1, \ldots, e_r \in M$, $b_{ij} \in A$ with $m_i = \sum b_{ij} e_j$, such that $\sum a_i b_{ij} = 0$.

   Compare this to a basis for a free module. If M is free, then we can choose our e_is from a basis for M.

4. Direct limits of flat modules are flat. This is because as in the previous step, any relation will live in a finite submodule of the limit.

   The same result applies for filtered colimits, but not for arbitrary colimits. This is because any module is a colimit of free modules. For example, $\mathbb{Z}/2\mathbb{Z}$, which is not flat as a $\mathbb{Z}$-module, can be written as the colimit of the diagram

   $$\mathbb{Z} \overset{0}{\underset{2}{\rightrightarrows}} \mathbb{Z}$$

5. Let $A \to B$ be a ring map, M a flat A-module, then $M \otimes_A B$ is a flat B-module.

6. M is A-flat if and only if $M_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-flat for all $\mathfrak{p} \in \operatorname{Spec} A$. So flatness is a local condition.

7. If B is a flat A-algebra, M a flat B-module, then M is a flat A-module.

8. If A is a field, then all A-modules are flat (because they are all vector spaces, which are free).

   If A is a PID, then M is A-flat if and only if M is A-torsion free. This can be seen by applying property (1) to a principal ideal.

9. If A is Noetherian, M a finite A-module, then M is flat if and only if M is projective, so $M \oplus M'$ is free for some M', if and only if $\tilde{M}$ is locally free on $\operatorname{Spec} A$, if and only if $M_{\mathfrak{p}}$ is free for all $\mathfrak{p} \in \operatorname{Spec} A$.

Here are some basic facts about flatness and cohomology.

Proposition 4.2. *Let*

$$\cdots \longrightarrow K^{n-1} \xrightarrow{d^{n-1}} K^n \xrightarrow{d^n} K^{n+1} \longrightarrow \cdots$$

be a complex of A-modules and A-linear maps (so that $d^n \circ d^{n-1} = 0$ for all n). Let M be an A-module. There is a natural homomorphism

$$H^n(K) \otimes_A M \to H^n(K \otimes_A M) \tag{4.1}$$

which is an isomorphism if M is A-flat.

Proof. We have that $H^n(K) = \ker d^n / \operatorname{im} d^{n-1}$. Then we have exact sequences

$$0 \longrightarrow \ker d^n \longrightarrow K^n \xrightarrow{d^n} K^{n+1}$$

and by the functoriality of $\_\_ \otimes M$ we get a morphism

$$\sigma : \ker d^n \otimes_A M \to \ker(d^n \otimes \mathrm{id}). \tag{4.2}$$

We also have an exact sequence

$$K^{n-1} \xrightarrow{d^{n-1}} \ker d^n \longrightarrow H^n(K) \longrightarrow 0$$

Tensoring preserves exactness:

$$\begin{array}{ccccccc}
K^{n-1} \otimes M & \longrightarrow & \ker d^n \otimes M & \longrightarrow & H^n(K) \otimes M & \longrightarrow & 0 \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \lambda} & & \\
K^{n-1} \otimes M & \longrightarrow & \ker(d^n \otimes \mathrm{id}) & \longrightarrow & H^n & \longrightarrow & 0
\end{array}$$

The map λ exists because the rows are exact and the other two vertical arrows exist. If M is flat, then σ is an isomorphism, so λ is also an isomorphism. □

Definition 4.3. Let $f : X \to Y$ be a morphism of schemes, $x \in X$. Then f is *flat* at x if $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-algebra and f is flat if it is flat at every $x \in X$.

A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is flat over Y if $\mathcal{F}_x$ is $\mathcal{O}_{Y,f(x)}$-flat for all $x \in X$. So $f : X \to Y$ is flat if $\mathcal{O}_X$ is flat over Y.

If $Y = X$, $f : X \to X$ the identity, then if $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module is flat over X, we say it is a flat $\mathcal{O}_X$-module.

1. $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat if and only if B is A-flat.

2. Open immersions are flat because $\mathcal{O}_{X,x} = \mathcal{O}_{Y,f(x)}$.

3. Closed immersions are rarely flat because A/I typically has torsion as an A-module.

4. If X, Y are k-schemes, then $X \otimes_k Y \to Y$ is flat because $A \otimes_k B$ is flat over B as A is k-flat.

Remark 4.4. Morally, a flat morphism can be thought of as a continuously varying family. Read Hartshorne III.9.

Proposition 4.5. *Let $f : X \to Y$ be a morphism of integral schemes of finite type over k. If f is flat, then for all closed points $y \in Y$, and for all irreducible components $Z \subset f^{-1}(y) \subset X$ of the fiber at y, then*

$$\dim Z + \dim Y = \dim X. \tag{4.3}$$

So a flat morphism is like a projection. Since $X \times_k Y \to Y$ is flat, and we have $f^{-1}(y) \cong X$, we get $\dim X + \dim Y = \dim X \times_k Y$.

Example 4.6. Let $f : \mathbb{A}^2_k \to \mathbb{A}^2_k$ be given in coordinates by $(u, v) \mapsto (u, uv)$. Then if $u \neq 0$ the fiber of f above (u, w) is $(u, w/u)$. But $f^{-1}(0, 0)$ is a line, so f is not flat (blowup?).

Example 4.7. Although fiber dimension is constant in a flat family, other things can vary. Let $Y = \operatorname{Spec} k[t] = \mathbb{A}^1_k$ and consider the injections into $X_1 = \operatorname{Spec} k[x, y, t]/(xy - t)$ and $X_2 = \operatorname{Spec} k[x, y, t]/(x^2 - ty)$. The fibers over $t = 1$ are $xy = 1$, and $y = x^2$, respectively, which are nice and integral. But the fiber of X_1 over $t = 0$ is $xy = 0$, which is not reduced, and the fiber of X_2 over $t = 0$ is $x^2 = 0$, which is reducible. But note that the fibers all have the same dimension.

Example 4.8. Let $f : X \to Y = \operatorname{Spec} A$ with A a PID (or a Dedekind domain). Then f is flat if and only if $\forall x \in X$, $\mathcal{O}_{X,x}$ is A-torsion free. So f is flat and X is integral with generic point $\eta \in X$, then $\mathcal{O}_{X,\eta}$ is A-torsion free, so $f(\eta)$ is the generic point of $\operatorname{Spec} A$. So if f is proper, then f is surjective, and in general it is dominant. A morphism is *dominant* if the image is a dense subset.

If Y is a regular scheme, then the converse of our dimension result is often true.

Theorem 4.9. *Let $f : X \to Y$ be a morphism of integral schemes of finite type over k, Y regular. If X is also regular (this condition can be weakened), and if for all $y \in Y$, and for all $Z \subset f^{-1}(y)$ an irreducible component of the fiber, we have that $\dim X = \dim Z + \dim Y$, then f is flat.*

Example 4.10. Y being regular in the above example is essential. Let $Y = \operatorname{Spec} k[u, v]/(u^3 - v^2) = \operatorname{Spec} A$ with $\operatorname{char} k \neq 2, 3$. And let $X \to Y$ be the normalization, so $X = \operatorname{Spec} k[t] \cong \mathbb{A}^1$ with the parametrization $u = t^2$, $v = t^3$. Then f is finite. So if f were flat, then $k[t]$ would be a locally free module over $k[u, v]/(u^3 - v^2)$. But for $\mathfrak{m} \subset A$ a maximal ideal, we have that $\dim A/\mathfrak{m}k[t]/\mathfrak{m}k[t]$ is 2 if $\mathfrak{m} = (u, v)$ because $k[t]/\mathfrak{m}k[t] = k[t]/(t^2)$, and 1 otherwise.

Remark 4.11. For X to be flat, it's enough to assume that X is locally a complete intersection, so that locally X looks like $\operatorname{Spec} k[t_1, \ldots, t_{m+\dim X}]/(f_1, \ldots, f_m)$. This if call "miracle flatness", Vakil talks a lot about it.

Here's one final property of flat morphisms, which is fairly hard to prove.

Lemma 4.12. *Let $f : X \to Y$ be a flat morphism which is locally of finite presentation (for instance f is of finite type, and Y is Noetherian). Then f is an open map, so it takes open sets to open sets.*

Example 4.13. $\operatorname{Spec}\mathbb{Q} \to \operatorname{Spec}\mathbb{Z}$ is flat but not open.

For more on flat morphisms, read III.9 of Hartshorne, especially the particularly illuminating example III.9.8.4. But don't use the first edition as there is a mistake.

The nicest sort of family is a family with smooth fibers.

Definition 4.14. Let $f : X \to Y$ be a locally of finite presentation, $x \in X$, $y = f(x) \in Y$. Then f is *smooth over* x (of relative dimension d) if it is flat over x, and there exists an open $x \in U \subset X_y = f^{-1}(y)$ such that U is smooth at x over $f^{-1}(y)$ (of dimension d).

We say f is *smooth* if it is smooth at every $x \in X$ (if and only if f is flat, and the connected components of every fiber X_y are smooth over $k(y)$).

Example 4.15. Y, Z varieties over k, Z smooth over k. Then $X = Y \times_k Z \to Y$ the projection map is smooth.

Theorem 4.16. *$f : X \to Y$ locally of finite presentation. TFAE:*

1. *f is smooth (as defined above).*

2. *f is flat, and $\Omega_{X/Y}$ is free of rank d (the fiber dimension, see below).*

3. *$\forall x \in X$, there exists an open affine neighborhood $x \in U = \operatorname{Spec} B \subset X$ such that $f(U) \subset V = \operatorname{Spec} A \subset Y$, with $B = A[t_1, \ldots, t_{m+d}]/(f_1, \ldots, f_m)$, such that the matrix*

$$\left(\frac{\partial f_i}{\partial t_j}(x)\right)_{ij} \in \operatorname{Mat}_{m+d,m}(k(x)) \tag{4.4}$$

has rank m.

The second condition means that for any connected component $Y' \subset Y$, $\Omega_{X/Y}$ is locally free of rank d on $f^{-1}(Y')$, where d is the dimension of every irreducible component of $f^{-1}(y)$ for every $y \in Y'$.

A smooth morphism is a family of smooth schemes of dimension d. When $d = 0$, a smooth morphism is then intuitively a "local isomorphism".

Definition 4.17. A morphism f is *etale* if it is smooth of relative dimension 0. In other words, f is locally of the form $\operatorname{Spec} B \to \operatorname{Spec} A$, where

$$B = A[t_1, \ldots, t_m]/(f_1, \ldots, f_m), \tag{4.5}$$

with $\det\left(\frac{\partial f_i}{\partial t_j}\right)_{ij}$ is invertible.

This is sort of an analog of the implicit function theorem: locally, it looks like a function.

Example 4.18. Let $\operatorname{char} k \neq 2$, and consider $k[u, v]/(u - v^2) \to k[u]$ sending $v \to v^2 = u$. Away from $u = v = 0$, f is etale.

f is etale if it is locally of finite presentation and flat and unramified, so $\Omega_{X/Y} = 0$.

5 Sheaf cohomology

In this section, all schemes will be separated and Noetherian, unless otherwise stated. So our schemes will be quasicompact and the intersections of affines are affine.

5.1 Homological algebra

A *complex* $(A^\bullet, d^\bullet)$ is a chain

$$A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \longrightarrow \cdots$$

of R-modules or sheaves, so $d^{p+1} \circ d^p = 0$ for all p.

We have cohomology groups $H^n(A^\bullet) = H^n(A^\bullet, d^\bullet) = \ker d^n / \operatorname{im} d^{n-1}$, and we define the direct sum

$$H^*(A^\bullet) = \bigoplus_{n \geq 0} H^n(A^\bullet) \tag{5.1}$$

which has the obvious grading.

Complexes form a category, with a morphism $(A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet)$ being a family of maps $f^p : A^p \to B^p$ commuting with d, so that $fd = df$. This induces a map $H^p(f) : H^p(A) \to H^p(B)$.

It is a basic fact of cohomology, that if we have an exact sequence

$$0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0$$

we get a long exact sequence in cohomology

$$0 \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to \cdots \tag{5.2}$$

which is functorial with respect to exact sequences, so if we have a commuting diagram between exact sequences we get a morphism of cohomology complexes.

Definition 5.1. Suppose we have $(A^\bullet, d_A^\bullet)$ and $(B^\bullet, d_B^\bullet)$ complexes and we are given maps $h^p : A^p \to B^{p-1}$ for all p, where $h^0 = 0$. If $h \circ d + d \circ h$ is a morphism of complexes which induces the zero map from $H^p(A) \to H^p(B)$, then we say that h is a *homotopy*.

If $f, g : A^\bullet \to B^\bullet$ are maps of complexes such that $f - g = dh + hd$, then f, g induces the same map $H^p(A) \to H^p(B)$ on cohomology. Then h is then called a homotopy between f, g and we say that f, g are *homotopic*.

In particular, if we have maps $h^p : A^p \to A^{p-1}$ such that $hd + dh = \mathrm{id}_A$, then we have that $H^p(\mathrm{id}_A) = 0$, so $H^p(A) = 0$ for all p. We then say that $A^\bullet$ is *null-homotopic*

If $A^\bullet$ is null-homotopic, then $H^*(A) = 0$. For vector spaces over a field k, the converse is true. Suppose $H^p(A^\bullet) = 0$ for all $p \geq 0$. By linear algebra, we have that our complex is of the form

$$B^0 = A^0 \to B^0 \oplus B^1 = A^1 \to B^1 \oplus B^2 = A^2 \to \cdots \tag{5.3}$$

with the obvious projection maps, and we can define $h^p : A^p \to A^{p-1}$ as the obvious maps in the other direction. Then h is a null homotopy.

Note that this converse is not true for abelian groups: the complex

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

has zero cohomology but is not null homotopic.

Remark 5.2. Let us work over complexes of R-modules. For $p \geq 0$, let $R[-p]$ be the complex which is R in degree p, and 0 elsewhere (in general, $[-p]$ means to shift $-p$ places to the left).

Then for any complex $A^\bullet$ of R-modules, $H^p(A)$ equals the homotopy classes of morphisms of complexes $R[-p] \to A^\bullet$. We can see this as follows:

$$\begin{array}{ccccc} 0 & \longrightarrow & R & \longrightarrow & 0 \\ & \overset{h}{\swarrow} & \downarrow f & \overset{0}{\searrow} & \\ A^{p-1} & \xrightarrow{d^{p-1}} & A^p & \xrightarrow{d^p} & A^{p+1} \end{array}$$

Our morphism f is determined by sending $1 \to a \in A$, and since the composite with d^p is zero, we must have $a \in \ker d^p$. Now, if we have a homotopy map h, h is determined by sending $1 \to a' \in A^{p-1}$. Then the $d^{p-1}(a') \in \operatorname{im} d^{p-1}$. If f and g are homotopic, then $f - g = dh + hd = dh$ because $hd = 0$, so f and g differ by a map sending $1 \to a'' \in \operatorname{im} d^{p-1}$. Thus the set of morphisms modulo homotopy is $\ker d^p / \operatorname{im} d^{p-1} = H^p(A)$.

Lemma 5.3 (Snake lemma)**.** *Suppose we have a commutative diagram with exact rows*

$$\begin{array}{ccccccc} & & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & & \end{array}$$

Then we get a long exact sequence

$$\ker f \to \ker g \to \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h \tag{5.4}$$

and we have that $\ker f = H^0(A' \to A)$ *and* $\operatorname{coker} f = H^1(A' \to A)$ *for example.*

5.2 Tensor product of complexes

Let $(A^\bullet, d_A^\bullet)$ and $(B^\bullet, d_B^\bullet)$ be complexes of R-modules. Then we have a tensor product complex $(A^\bullet \otimes_R B^\bullet, d_{A\otimes B})$ which in degree n is

$$\bigoplus_{p+q=n} A^p \otimes_R B^q \tag{5.5}$$

with differential

$$d(a \otimes b) = da \otimes b + (-1)^p a \otimes db \tag{5.6}$$

if $a \in A^p$, $b \in B^q$. Under this differential map $A^p \otimes B^q \mapsto (A^{p+1} \otimes B^q) \oplus (A^p \otimes B^{q+1})$. What is the cohomology of this complex?

Theorem 5.4 (Naive Künneth formula)**.** *Let $R = k$ be a field. Then*

$$H^n(A^\bullet \otimes_k B^\bullet) = \bigoplus_{p+q=n} H^p(A^\bullet) \otimes_k H^q(B^\bullet). \tag{5.7}$$

Proof. Let $H^\bullet(A)$ be the complex

$$H^0(A^\bullet) \xrightarrow{\quad 0 \quad} H^1(A^\bullet) \xrightarrow{\quad 0 \quad} \cdots$$

Then since k is a field, we can split off each $H^p(A)$ as a summand of A^p:

$$A^\bullet \cong H^\bullet(A) \oplus^\bullet \tag{5.8}$$

where $C^\bullet$ has zero cohomology. Essentially we split off the "cohomology" part of A, and are left with the complex $C^\bullet$, and the boundary operators are all exact (which is easy to see if you write it out), so they have zero cohomology.

Since $C^\bullet$ is a vector space with zero cohomology, it is null homotopic, so there exists $h^p : C^p \to C^{p-1}$ with $d_C h + hdC = \operatorname{id} C$. Then

$$A^\bullet \otimes_k B^\bullet \cong H^\bullet(A) \otimes_k B^\bullet \oplus C^\bullet \otimes_k B^\bullet \tag{5.9}$$

and $h' = h \otimes \operatorname{id}_B$ is a null homotopy for the complex $C^\bullet \otimes_k B^\bullet$ (we can check that $h' \circ d_{C\otimes B} + d_{C\otimes B} \circ h' = \operatorname{id}_{C\otimes B}$). Thus $C^\bullet \otimes_k B^\bullet$ has zero cohomology.

Then

$$\begin{aligned} H^n(A^\bullet \otimes B^\bullet) &= H^n(H^\bullet(A) \otimes B^\bullet) \\ &= H^n(H^\bullet(A) \otimes H^\bullet(B) \\ &= \bigoplus_{p+q=n} H^p(A) \otimes H^q(B). \end{aligned} \tag{5.10}$$

□

5.3 Sheaf cohomology

Recall that if $\mathcal{F}$ is a sheaf of abelian groups on a topological space X, then sheaf cohomology gives groups $H^p(X, \mathcal{F})$ such that

$$H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F}) = \mathcal{F}(X) \tag{5.11}$$

and for all short exact sequences of sheaves

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0 \tag{5.12}$$

we get a long exact sequence

$$0 \to H^0(\mathcal{F}_1) \to H^0(\mathcal{F}_2) \to H^0(\mathcal{F}_3) \to H^1(\mathcal{F}_1) \to \cdots \tag{5.13}$$

which "remedies" the fact that $H^0(X, \mathcal{F}_2) \to H^0(X, \mathcal{F}_3)$ may not be surjective. This is because the functor $\Gamma(X, \_\_)$ is left exact but not right exact.

We can define sheaf cohomology with injective resolutions, but this is not very concrete as giving descriptions of injective modules is difficult. But we can also compute sheaf cohomology in a few other ways.

First, recall that a sheaf $\mathcal{G}$ is flasque (or flabby) if for all opens $U \subset V \subset X$, the restriction map $\rho_{UV} : \mathcal{G}(V) \to \mathcal{G}(U)$ is surjective. Then if

$$0 \to \mathcal{F} \to \mathcal{G}^0 \to \mathcal{G}^1 \to \cdots \tag{5.14}$$

is a long exact sequence with $\mathcal{G}^p$ flasque, then $H^p(X,\mathcal{F}) = H^p(\Gamma(X,\mathcal{G}^\bullet))$. So the cohomology of $\mathcal{F}$ is given by the cohomology of the global sections of $\mathcal{G}^\bullet$. In particular, if $\mathcal{F}$ is flasque, we can set $\mathcal{G}^0 = \mathcal{F}$ and $\mathcal{G}^p = 0$ for $p \geq 1$, and we get that $H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F})$ and $H^p(X,\mathcal{F}) = 0$ if $p \geq 1$.

We can generalize this method to acyclic sheaves: recall that $\mathcal{G}$ is an acyclic sheaf if for all $p > 0$, then $H^p(X,\mathcal{G}) = 0$. Then if

$$0 \to \mathcal{F} \to \mathcal{G}^0 \to \mathcal{G}^1 \to \cdots \tag{5.15}$$

with $\mathcal{G}^p$ acyclic, then $H^p(X,\mathcal{F}) = H^p(\Gamma(X,\mathcal{G}^\bullet))$. Every sheave $\mathcal{F}$ has a canonical resolution of the form (5.15) by flasque sheaves, called the *Godemont resolution.* Let $G\mathcal{F}$ be the sheaf

$$G\mathcal{F} = \prod_{x\in X} (i_x)_*(\mathcal{F}_x), \tag{5.16}$$

where $i_x : \{x\} \hookrightarrow X$ is the inclusion. There is a canonical injection $\mathcal{F} \to G\mathcal{F}$ by $\mathcal{F}_x \to (i_x)_*\mathcal{F}_x$ and $G\mathcal{F}$ is clearly flasque by construction, because we have

$$G\mathcal{F}(U) = \prod_{x\in U} \mathcal{F}_x. \tag{5.17}$$

We then build a complex

$$0 \longrightarrow \mathcal{F} \xrightarrow{a^0} \mathcal{G}^0 = G\mathcal{F} \xrightarrow{a^1} \mathcal{G}^1 = G(\operatorname{coker} a^0) \xrightarrow{a^2} \mathcal{G}^2 = G(\operatorname{coker} a^1) \longrightarrow \cdots$$

Sheaf cohomology is functorial in the following sense. Let $f : X \to Y$ be a continuous map, and let $\mathcal{F}$ be a sheaf on X and $\mathcal{G}$ a sheaf on Y. Then for any map $\varphi : G \to f_*\mathcal{F}$ (which is the same as giving a map $f^*\mathcal{G} \to \mathcal{F}$), there is an induced map

$$H^p(Y,\mathcal{G}) \to H^p(X,\mathcal{F}). \tag{5.18}$$

Proposition 5.5. *Let $i : Y \to X$ be a closed subset, and $\mathcal{F}$ a sheaf on Y. Then for all p,*

$$H^p(Y,\mathcal{F}) = H^p(X, i_*\mathcal{F}) \tag{5.19}$$

Proof. Choose a flasque resolution on Y

$$0 \to \mathcal{F} \to \mathcal{G}^0 \to \mathcal{G}^1 \to \cdots \tag{5.20}$$

Then $H^*(Y,\mathcal{F}) = H^*(\Gamma(Y,\mathcal{G}^\bullet))$.

By definition, $i_*\mathcal{G}^n$ is a flasque sheaf on X, so $H^p(X, i_*\mathcal{F}) = H^p(\Gamma(X, i_*\mathcal{G}^\bullet)) = H^p(\Gamma(Y,\mathcal{G}^\bullet)) = H^*(Y,\mathcal{F})$ as well. ☐

Theorem 5.6 (Mayer- Vietoris sequence)**.** *Let $U, V \subset X$ be open sets with $U \cup V = X$, $\mathcal{F}$ a sheaf on X. Then we have a long exact sequence*

$$0 \longrightarrow H^0(X,\mathcal{F}) \longrightarrow H^0(U,\mathcal{F}|_U) \oplus H^0(V,\mathcal{F}|_V) \xrightarrow{r^0} H^0(U\cap V,\mathcal{F}|_{U\cap V}$$

$$\longrightarrow H^1(X,\mathcal{F}) \longrightarrow H^1(U,\mathcal{F}|_U) \oplus H^1(V,\mathcal{F}|_V) \xrightarrow{r^1} H^1(U\cap V,\mathcal{F}|_{U\cap V}) \longrightarrow \cdots$$

where r^p is the difference of the restriction maps induced by $U \cap V \hookrightarrow U, V$, so that $r^p = \rho_{U,U\cap V} - \rho_{V,U\cap V}$.

Proof. Let $0 \to \mathcal{F} \to G^0 \to G^1 \to \cdots$ be a flasque resolution of $\mathcal{F}$. Then

$$0 \longrightarrow \Gamma(X, \mathcal{G}^\bullet) \longrightarrow \Gamma(U, \mathcal{G}^\bullet) \oplus \Gamma(V, \mathcal{G}^\bullet) \xrightarrow{r^0} \Gamma(U \cap V, \mathcal{G}^\bullet) \longrightarrow 0$$

is an exact sequence of complexes of groups (it is left exact by the sheaf axioms, and right exact as $\mathcal{G}^\bullet$ are all flasque).

As $H^*(\_\_, \mathcal{F}) = H^*(\Gamma(\_\_, \mathcal{G}^\bullet))$ for $\_\_ \in \{X, U, V, U \cap V\}$, then the long exact sequence for cohomology gives the result (take the cohomology of the exact sequence). □

Now assume that X is a scheme which is separated and Noetherian. It is a basic fact that for any sheaf $\mathcal{F}$ on X, if $\dim X = d$, then $H^p(X, \mathcal{F}) = 0$ for all $p > d$.

For quasicoherent sheaves of $\mathcal{O}_X$-modules, we can compute H^* using Čech cohomology Let $X = \bigcup_{i=1}^N U_i$ be an open cover, $\mathcal{F}$ a sheaf on X. Then we define

$$\check{H}^*((U_i), \mathcal{F}) = H^*(\check{C}((U_i), \mathcal{F}), \check{d}), \tag{5.21}$$

where

$$\check{C}^p((U_i), \mathcal{F}) = \prod_{1 \le i_0 < \cdots < i_p \le N} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p}) \tag{5.22}$$

and

$$\begin{aligned} \check{d}^p : \check{C}^p((U_i), \mathcal{F}) &\to \check{C}^{p+1} \\ s = (s_{i_0 < \cdots < i_p}) &\mapsto t = (t_{i_0 < \cdots < i_{p+1}}) \end{aligned} \tag{5.23}$$

where

$$t_{i_0 < \cdots < i_{p+1}} = \sum_{\alpha=0}^{p+1} (-1)^\alpha s_{i_0 < \cdots (\check{i}_\alpha) \cdots < i_{p+1}}|_{U_{i_0} \cap \cdots \cap U_{i_{p+1}}} \tag{5.24}$$

where $(\check{i}_\alpha)$ means omit α.

Convenient notation: if $I = \{i_0 < \cdots < i_p\}$, then $U_I = U_{I_0} \cap \cdots \cap U_{i_p}$. So

$$\check{C}^p((U_i), \mathcal{F}) = \prod_{\#I = p+1} \mathcal{F}(U_I). \tag{5.25}$$

There exists alternative, equivalent definitions of Čech cohomology which don't rely on the ordering of the open cover.

Remark 5.7. Assume that $\mathcal{F}$ is a quasicoherent sheaf of $\mathcal{O}_X$-modules, and that (U_i) is an open affine covering, so that all the U_Is are affine as X is separated. Then $H^*(X, \mathcal{F}) \cong \check{H}^*((U_i), \mathcal{F})$, and this isomorphism is canonical.

In particular, if X is affine, and $\mathcal{F} = \tilde{M}$, taking $(U_i) = (X)$, we get that $H^0(X, \mathcal{F}) = M$, and $H^p(X, \mathcal{F}) = 0$ for $p > 0$.

A more general principle is that if $\mathcal{F}$ has no cohomology in degree 0 on all of the U_Is (so that $\mathcal{F}|_{U_I}$ is acyclic), then Čech cohomology is the same as sheaf cohomology. We can think of this fact as a sort of generalization of the Mayer-Vietoris sequence.

Lemma 5.8. *Let $j : V \hookrightarrow X$ be an inclusion of an open affine, and $\mathcal{F}$ a quasicoherent $\mathcal{O}_V$-module. Then $j_*\mathcal{F}$ is quasicoherent and is acyclic.*

Proof. X is separated, which implies that j is an affine map, as if $U \subset X$ is affine, then $j^{-1}(U) = U \cap V$ is affine.

By Question 4 of Example sheet 1, we have that if $j_*\mathcal{F}$ is quasicoherent, we can compute its Čech cohomology:

$$\check{C}^p((U_i), j_*\mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p} \cap V) = \check{C}^p((U_i'), \mathcal{F}) \tag{5.26}$$

where if $X = \bigcup U_i$, then $U_i' = V \cap U_i$. Then

$$\check{H}^p((U_i), j_*\mathcal{F}) = \check{H}^p((U_i'), \mathcal{F}) = 0 \tag{5.27}$$

if $p > 0$ as V is affine, and the two cohomology groups are equal because the associated Čech complexes are equal. □

We can also develop a "sheaf version" of Čech cohomology: for $V \subset X$ open, define

$$\check{\mathcal{C}}^\bullet((U_i), \mathcal{F})(V) = \check{C}^\bullet((U_i \cap V), \mathcal{F}|_V) \tag{5.28}$$

so that

$$\check{\mathcal{C}}^p((U_i), \mathcal{F}) = \prod_{\#I = p+1} (j_I)_*(\mathcal{F}|_{U_I}) \tag{5.29}$$

where $j_I : U_I \hookrightarrow X$ is the inclusion. This makes it clear that $\check{\mathcal{C}}$ is in fact a sheaf, as it is a product of the pushforward sheaves $(j_I)_*(\mathcal{F}|_{U_I})$.

The Čech differentials give a complex of sheaves

$$0 \to \mathcal{F} \to \check{\mathcal{C}}^0((U_i), \mathcal{F}) \to \check{\mathcal{C}}^1((U_i), \mathcal{F}) \to \cdots \tag{5.30}$$

Proposition 5.9. *Assume (U_i) is an affine cover and $\mathcal{F}$ is quasicoherent. Then* (5.30) *is an exact sequence of quasicoherent sheaves and*

$$H^n(X, \check{\mathcal{C}}^p((U_i), \mathcal{F})) = \begin{cases} \check{C}^p((U_i), \mathcal{F}) & n = 0 \\ 0 & n > 0 \end{cases} \tag{5.31}$$

Proof. By Lemma 5.8, applied to $j_{I_*}(\mathcal{F}|_{U_I})$, the sheaves $\check{\mathcal{C}}^p$ are quasicoherent and acyclic for all $p \geq 0$ because they are the product of quasicoherent acyclic sheaves. We have that

$$H^0(X, \check{\mathcal{C}}^p((U_i), \mathcal{F}) = \Gamma(X, \check{\mathcal{C}}^p((U_i), \mathcal{F})) = \prod_{\#I = p+1} \mathcal{F}(U_I) = \check{C}^p((U_i), \mathcal{F}). \tag{5.32}$$

It remains to show the sequence (5.30) is exact. Let $V \subset X$ be an open affine. Then the sections over V of (5.30) are

$$0 \to \mathcal{F}(V) \to \check{\mathcal{C}}^0((U_i), \mathcal{F})(V) \to \check{\mathcal{C}}^1((U_i), \mathcal{F})(V) \to \cdots \tag{5.33}$$

and since $\check{\mathcal{C}}^p((U_i), \mathcal{F})(V) = \check{C}^p((U_i \cap v), \mathcal{F}|_V)$, we have that the cohomology of (5.30) is $H^*(V, \mathcal{F}|_V)$ which is acyclic because V is affine. Thus (5.30) is exact. □

Theorem 5.10 (Kunneth formula)**.** *Let X, Y be Noetherian, separated k-schemes for k a field, and let $\mathcal{F}$, $\mathcal{G}$ be quasicoherent sheaves of $\mathcal{O}_X$, $\mathcal{O}_Y$-modules, respectively. Then*

$$H^n(X \times_k Y, \operatorname{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_k Y}} \operatorname{pr}_2^* \mathcal{G}) \cong \bigoplus_{p+q=n} H^p(X, \mathcal{F}) \otimes_k H^q(Y, \mathcal{G}) \tag{5.34}$$

We write

$$\mathcal{F} \boxtimes \mathcal{G} \coloneqq \operatorname{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_k Y}} \operatorname{pr}_2^* \mathcal{G} \tag{5.35}$$

for the "product" of $\mathcal{F}$ and $\mathcal{G}$ as a $\mathcal{O}_{X \times_k Y}$-module.

We will use a preparatory lemma.

Lemma 5.11. *(i) Let A, B be k-algebras, M an A-module, N a B-module. Then on $\operatorname{Spec} A \times_k \operatorname{Spec} B = \operatorname{Spec} A \otimes_k B$, we have that*

$$\tilde{M} \boxtimes \tilde{N} = \widetilde{M \otimes_k N} \tag{5.36}$$

(ii) If $\mathcal{F}$, $\mathcal{G}$ are as in the theorem, and $\mathcal{F} \to \mathcal{K}^\bullet$ and $\mathcal{G} \to \mathcal{L}^\bullet$ are resolutions of quasicoherent sheaves, then

$$\mathcal{F} \boxtimes \mathcal{G} \to \mathcal{K}^\bullet \boxtimes \mathcal{L}^\bullet \tag{5.37}$$

is a resolution, where

$$(\mathcal{K}^\bullet \boxtimes \mathcal{L}^\bullet)^n = \bigoplus_{p+q=n} \mathcal{K}^p \boxtimes \mathcal{L}^q \tag{5.38}$$

Proof. (i) Let $\mathcal{F} = \tilde{M}$. Then $\operatorname{pr}_1^* \mathcal{F} = \widetilde{M \otimes_A (A \otimes_k B)}$ so

$$\tilde{M} \boxtimes \tilde{N} = \widetilde{(M \otimes_A (A \otimes_k B)) \otimes_{A \otimes_k B} ((A \otimes_k B) \otimes_B N)} = \widetilde{M \otimes_k N}. \tag{5.39}$$

We could make the tildes wider if we wanted to, but the question is, should we?

(ii) It is enough to check that this is true for an open affine covering, because being a complex is the same as $d^2 = 0$, and we can check being zero on an open affine covering. So assume that X, Y are affine, $\mathcal{F} = \tilde{M}$, $\mathcal{G} = \tilde{N}$, $\mathcal{K}^p = \tilde{K^p}$, and $\mathcal{L}^p = \tilde{L^p}$ with $M \to K^\bullet$, $N \to L^\bullet$ resolutions of groups/modules. It suffices to show (by part (i)) that $M \otimes N \to K^\bullet \otimes_k L^\bullet$ is a resolution, in other words that

$$H^n(K^\bullet \otimes_k L^\bullet) = \begin{cases} M \otimes_k N & n = 0 \\ 0 & n > 0 \end{cases} \tag{5.40}$$

But by the naive Kunneth formula Theorem 5.4, we have that

$$H^n(K^\bullet \otimes L^\bullet) = \bigoplus_{p+q=n} H^p(K^\bullet) \otimes H^q(L^\bullet) = \begin{cases} M \otimes_k N & n = 0 \\ 0 & n > 0 \end{cases} \tag{5.41}$$

since we assume that X, Y are affine. □

Proof of Theorem 5.10. Take affine open cover (U_i), (V_j) for X, Y. Then we have Čech resolutions by quasicoherents:

$$\mathcal{F} \to \check{\mathcal{C}}^\bullet((U_i), \mathcal{F}) \quad , \mathcal{G} \to \check{\mathcal{C}}^\bullet((V_j), \mathcal{G}). \tag{5.42}$$

So by Lemma 5.11 part (ii), we have a resolution

$$\mathcal{F} \boxtimes \mathcal{G} \to \mathcal{K}^\bullet \coloneqq \check{\mathcal{C}}^\bullet((U_i), \mathcal{F}) \boxtimes \check{\mathcal{C}}^\bullet((V_j), \mathcal{G}) \tag{5.43}$$

and we have that

$$\mathcal{K}^n = \bigoplus_{p+q=n} \prod_{\substack{\#I=p+1 \\ \#J=q+1}} \mathcal{K}_{I,J} \tag{5.44}$$

where

$$\begin{aligned} K_{I,J} &= (j_I)_* \mathcal{F}|_{U_I} \boxtimes (j_J)_* \mathcal{G}|_{V_J} \\ &= (j_I \times j_J)_*((\mathcal{F} \boxtimes \mathcal{G})|_{U_I \times V_J}). \end{aligned} \tag{5.45}$$

As each $j_I \times j_J : U_I \times V_j \hookrightarrow X \times Y$ is an inclusion of open affines, by Lemma 5.8, $\mathcal{K}^n$ is acyclic. So $H^*(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = H^*(\Gamma(X \times Y, \mathcal{K}^\bullet))$, and we have that

$$\begin{aligned} \Gamma(X \times Y, \mathcal{K}_{I,J}) &= \Gamma(U_I \times V_J, \mathcal{F} \boxtimes \mathcal{G}) \\ &= \mathcal{F}(U_I) \otimes_k \mathcal{G}(V_J) \end{aligned} \tag{5.46}$$

by Lemma 5.11 (i), so

$$\Gamma(X \times Y, \mathcal{K}^\bullet) = \check{\mathcal{C}}^\bullet(U, \mathcal{F}) \otimes_k \check{\mathcal{C}}^\bullet(V, \mathcal{G}) \tag{5.47}$$

and by the naive Kunneth formula Theorem 5.4 we have that the cohomology of this complex is

$$\bigoplus_{p+q=n} H^p(X, \mathcal{F}) \otimes_k H^q(Y, \mathcal{G}) \tag{5.48}$$

in degree n. □

The Kunneth formula gives rise to the cup product on cohomology as follows. Let $Y = X$, and $\Delta : X \hookrightarrow X \times_k X$ be the diagonal embedding. Let $\mathcal{F}, \mathcal{G}$ be quasicoherent sheaves on X. Then $\Delta^*(\mathcal{F} \boxtimes \mathcal{G}) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$. The cup product is the composite map

$$H^p(X, \mathcal{F}) \otimes_k H^q(X, \mathcal{G}) \xhookrightarrow{\text{Kunneth}} H^{p+q}(X \times_k X, \mathcal{F} \boxtimes \mathcal{G}) \xrightarrow{\Delta^*} H^{p+q}(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}).$$

If we take $\mathcal{F} = \mathcal{G} = \mathcal{O}_X = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$, then

$$H^*(X, \mathcal{O}_X) = \bigoplus_{n \geq 0} H^n(X, \mathcal{O}_X) \tag{5.49}$$

becomes a k-algebra. It is a fact (the proof is omitted) that $H^*(X, \mathcal{O}_X)$ is a commutative-graded k-algebra:

$$x \smile y = (-1)^{pq} y \smile x \tag{5.50}$$

if $x \in H^p(X, \mathcal{O}_X)$ and $y \in H^q(X, \mathcal{O}_X)$.

5.4 Cohomology of projective spaces, and Hilbert polynomials

Let X be a Noetherian scheme, $\mathcal{F}$ a coherent $\mathcal{O}_X$-module (so it is locally $\tilde{M}$ for a finite module M).

Definition 5.12. The *support* of $\mathcal{F}$ is

$$\operatorname{supp}\mathcal{F} = \{x \in X | \mathcal{F}_x \neq 0\} \subset X. \tag{5.51}$$

This is a closed subset of X since if $X = \operatorname{Spec} A$, $\mathcal{F} = \tilde{m}$, then

$$\operatorname{supp}\mathcal{F} = \{\mathfrak{p} \in \operatorname{Spec} A, M_{\mathfrak{p}} \neq 0\} = V(\operatorname{ann} M) \tag{5.52}$$

because $M_{\mathfrak{p}} = 0$ if and only if there exists $a \in A \backslash \mathfrak{p}$ such that $aM = 0$ because M is finite if and only if $\operatorname{ann} M \not\subset \mathfrak{p}$.

More precisely: there exists a closed subscheme $i : Z \hookrightarrow X$ with point set $\operatorname{supp}(\mathcal{F})$ such that $\mathcal{F} = i_*\mathcal{G}$ for a coherent $\mathcal{O}_Z$-module $\mathcal{G}$ and Z is "as small as possible", so that if $\mathcal{F} = j_*\mathcal{G}'$ for a $\mathcal{O}_{Z'}$-module $\mathcal{G}'$, then $Z \subset Z'$. Note that Z may not be reduced. Also if $X = \operatorname{Spec} A$, then $\mathcal{F} = \tilde{M}$ and $Z = \operatorname{Spec} A/\operatorname{ann}(M)$.

Theorem 5.13 (Basic finiteness theorem)**.** *Let A be a Noetherian ring, $f : X \to \operatorname{Spec} A$ a proper morphism, so that X is Noetherian, separated, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $H^p(X, \mathcal{F})$ is a finite A-module.*

In particular, for a proper k-scheme X,

$$\dim_k H^p(X, \mathcal{F}) < \infty \tag{5.53}$$

and if $p > \dim X$, then $H^p(X, \mathcal{F}) = 0$, so $H^*(X, \mathcal{F})$ is finite dimensional.

This allows us to define the Euler characteristic.

Definition 5.14. Let X/k be a proper scheme, $\mathcal{F}$ a coherent sheaf. Then the Euler characteristic of $\mathcal{F}$ is

$$\chi(X, \mathcal{F}) = \sum_{p \geq 0} (-1)^p \dim_k H^p(X, \mathcal{F}) \in \mathbb{Z}. \tag{5.54}$$

Given a short exact sequence of sheaves

$$0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to 0, \tag{5.55}$$

it follows from the long exact sequence of cohomology

$$0 \to H^0(X, \mathcal{F}_0) \to H^0(X, \mathcal{F}_1) \to \cdots \tag{5.56}$$

that $\chi(X, \mathcal{F}_1) = \chi(X, \mathcal{F}_0) + \chi(X, \mathcal{F}_2)$. Recall that for all $n \in \mathbb{Z}$, there exists an invertible sheaf $\mathcal{O}_{\mathbb{P}}(n)$ on $\mathbb{P}^N_k$ whose sections on the standard open affine $U_j = \operatorname{Spec} k[T_0/T_j, \ldots, T_N/T_j] \subset \mathbb{P}^N_k$ are

$$\{f/T_j^d \mid f \text{ homogeneous of degree } n + d\} \tag{5.57}$$

and the transition functions are $(T_i/T_j)^n$. We have that

$$H^p(\mathbb{P}^N_k, \mathcal{O}_{\mathbb{P}}(n)) = \begin{cases} k[T_0, \ldots, T_N]_{\deg = n} & p = 0, n \geq 0 \\ k[T_0, \ldots, T_N]^{\vee}_{-N-n-1} & p = N, n \leq N - 1 \end{cases} \tag{5.58}$$

and the cohomology is 0 otherwise. In the first case we have that the cohomology group has dimension $\binom{N+n}{n}$ and in the second we have that it has dimension $\binom{-n-1}{N}$.

We then have that $\chi(\mathbb{P}^N_k, \mathcal{O}_{\mathbb{P}}(n)) = P(n)$, where

$$P(t) = \frac{(t+N)\cdots(t+1)}{N!} \in \mathbb{Q}[t] \tag{5.59}$$

is a polynomial. More generally we have the following

Theorem 5.15. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^N_k$. Then there exists $P(\mathcal{F}, t) \in \mathbb{Q}[t]$, the* Hilbert polynomial *of $\mathcal{F}$, such that $\forall n \in \mathbb{Z}$,*

$$\chi(\mathbb{P}^N, \mathcal{F}(n)) = P(\mathcal{F}, n) \tag{5.60}$$

where in general we write

$$\mathcal{F}(n) \coloneqq \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(n). \tag{5.61}$$

We have that $\deg P(\mathcal{F}, t) = \dim \operatorname{supp}(\mathcal{F})$.

Example 5.16. We have that

$$P(\mathcal{O}^n_{\mathbb{P}}, t) = \frac{(t+N)\cdots(t+1)}{N!}, \tag{5.62}$$

and $\deg P(\mathcal{O}^n_{\mathbb{P}}, t) = N = \dim \mathbb{P}^N_k$ as desired.

More generally, let X be a scheme which is projective over k, so there exists a closed immersion $i : X \hookrightarrow \mathbb{P}^N_k$ for some N, and define $\mathcal{O}_X(n) \coloneqq i^*\mathcal{O}_{\mathbb{P}}(n)$.

If X is projective, and $\mathcal{F}$ is coherent on X, set $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$. Then

$$H^*(X, \mathcal{F}(n)) \cong H^*(\mathbb{P}^N, (i_*\mathcal{F})(n)) \tag{5.63}$$

Thus if we define

$$P(X, \mathcal{F}, t) \coloneqq P(i_*\mathcal{F}, t), \tag{5.64}$$

then we have that

$$P(X, \mathcal{F}, n) = \chi(X, \mathcal{F}(n)) \tag{5.65}$$

for all $n \in \mathbb{Z}$.

Definition 5.17. For any k-scheme X and any invertible $\mathcal{O}_X$-module $\mathcal{L}$, say that $\mathcal{L}$ is *very ample* if there exists a closed immersion $i : X \to \mathbb{P}^N_k$ with $\mathcal{L} \cong \mathcal{O}_X(1)$, so that X is proper.

Now, let $\mathcal{L}$ be a very ample invertible $\mathcal{O}_X$-module. Then if $s_0, \ldots, s_N$ is a basis for $\Gamma(X, \mathcal{L})$, it determines a closed immersion $i : X \hookrightarrow \mathbb{P}^N$ with s_j equal to the pull back of $T_j \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(1))$.

Remark 5.18. An equivalent criterion to very ampleness if that $\mathcal{L}$ separates points and tangent vectors, we proved this on sheet 3 of algebraic geometry, it's in Hartshorne.

Now, we will define ampleness and give 4 equivalent criteria. First, we give a definition.

Definition 5.19. A sheaf $\mathcal{F}$ is *generated by global sections* if there exists a surjection $\mathcal{O}_X^{\oplus m} \twoheadrightarrow \mathcal{F}$.

Theorem 5.20. *Let X/k be a proper scheme, and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The following are equivalent:*

(i) For some $r \geq 1$, $\mathcal{L}^{\otimes r}$ is very ample.

(ii) There exists $r_0 \geq 1$, such that for all $r \geq r_0$, $\mathcal{L}^{\otimes r}$ is very ample.

(iii) for all coherent $\mathcal{O}_X$-modules $\mathcal{F}$, there exists $n_0 > 0$ such that for all $n \geq n_0$, $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n$ is generated by global sections.

(iv) (Serre's criterion) For all coherent $\mathcal{O}_X$-modules $\mathcal{F}$, there exists n_0 such that for all $n \geq n_0$ and for all $p > 0$, then

$$H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0. \tag{5.66}$$

If any of the above hold, we say that $\mathcal{L}$ is ample.

Remark 5.21. We add some intuition behind condition (iii) above. The sheaf $\mathcal{F}$ is coherent, so it has finite presentation, so it is isomorphic to $\mathcal{O}_X^{\oplus m}$ for some m, with some "denominators" coming from the presentation. We then have that $\mathcal{F}_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ "cancels" those denominators and makes $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ "projective".

Example 5.22. Let X be an elliptic curve over k, and $P \in X$ a k-rational point. Let $\mathcal{L} = \mathcal{O}_X(P)$. Then $\mathcal{L}^{\otimes 3}$ is very ample, because there is an embedding of X as a plane cubic in $\mathbb{P}^2$. But $\mathcal{L}$ is not very ample, so $\mathcal{L}$ is ample but not very ample.

In general, for a smooth proper curve X, $\mathcal{L}$ is ample if and only if $\mathcal{L} \cong \mathcal{O}_X(D)$, for D a divisor of degree > 0.

Thus ampleness can be thought of as a sort of "positivity condition".

The proof of all these results can be found in Hartshorne.

Proposition 5.23. *Let $X \subset \mathbb{P}^N_k$ be projective, integral, and* $\dim X = d$, *$\eta \in X$ the generic, and $\mathcal{F}$ a coherent sheaf of $\mathcal{O}_X$-modules. Then*

$$P(X, \mathcal{F}, t) = \left(\dim_{k(X)} \mathcal{F}_\eta\right) P(X, \mathcal{O}_X, t) + R(t), \tag{5.67}$$

where $R(t)$ has degree strictly less than d. Recall that $P(X, \mathcal{O}_X, t)$ has degree $\dim \operatorname{supp} \mathcal{O}_X = d$.

Proof. Let $e_1, \ldots, e_r$ be a $k(X)$-basis for $\mathcal{F}_\eta$, for some $r \geq 0$ (in particular, $\mathcal{F}_\eta$ could equal 0).

Then there exists n such that $e_1, \ldots, e_r$ extend to global sections of $\mathcal{F}(n)$ (why?). So we have an exact sequence

$$0 \to \mathcal{O}_X^{\oplus r} \to \mathcal{F}(n) \to \mathcal{G} \to 0 \tag{5.68}$$

where $\mathcal{G}$ is the cokernel. We have that $\mathcal{G}_\eta = 0$ because $\mathcal{G}_\eta = \mathcal{F}_\eta/(e_1, \ldots, e_r) = 0$ because $e_1, \ldots, e_r$ span $\mathcal{F}_\eta$. So $\eta \notin \operatorname{supp} \mathcal{G}$, so $\dim(\operatorname{supp} \mathcal{G}) < d$, and we have that

$$\begin{aligned} P(X, \mathcal{F}, t+n) &= P(X, \mathcal{F}(n), t) \\ &= P(X, \mathcal{G}, t) + P(X, \mathcal{O}_X^{\oplus r}, t) \\ &= P(X, \mathcal{G}, t) + rP(X, \mathcal{O}_X, t) \end{aligned} \tag{5.69}$$

and we are done because $P(X, \mathcal{G}, t)$ has degree less than d. □

Notation: we write $P(X, t) = P(X, \mathcal{O}_X, t)$, the *Hilbert polynomial of* X. Some people also sometimes write $P_X(\mathcal{F}, t) = P(X, \mathcal{F}, t)$.

5.5 Cohomology and base change

Let $f : X \to Y = \operatorname{Spec} A$ be a morphism of Noetherian, separated schemes with Y affine, and let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module. Let B be an A-algebra. The we have the fiber product base change diagram

$$\begin{array}{ccc} X_B \coloneqq X \times_A \operatorname{Spec} B & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} B & \xrightarrow{g} & \operatorname{Spec} A \end{array}$$

and we can define $\mathcal{F}_B = (g')^*\mathcal{F}$, which is a quasicoherent $\mathcal{O}_{X_B}$-module. By functoriality, there exists a map $H^p(X, \mathcal{F}) \to H^p(X_B, Y_B)$. This is an A-module homomorphism, and extending scalars gives a B-module homomorphism, called the base change homomorphism:

$$\beta_p : H^p(X, \mathcal{F}) \otimes_A B \to H^p(X_B, \mathcal{F}_B) \tag{5.70}$$

We are interested in finding out when β_p is an isomorphism. This is the same as the question: "When does cohomology commute with base change"?

Example 5.24. Let $s \in \operatorname{Spec} A$ be a point, $B = k(s) = \mathcal{O}_{Y,s}/\mathfrak{m}_s$. Then X_B is the fiber $f^{-1}(s) = X_s = X \times_A \operatorname{Spec} k(s)$ and we have an inclusion $i = g' : X_s \hookrightarrow X$, and $\mathcal{F}_B = i^*\mathcal{F} = \mathcal{F}(s)$. Thus we are asking when $H^p(X_s, \mathcal{F}(s)) = H^p(X, A) \otimes_A k(s)$. Even for $p = 0$, this need not hold.

Example 5.25. Let $X = \mathbb{A}^2_k \setminus \{0\} \to Y = \mathbb{A}^1_k = \operatorname{Spec} k[t]$ with the map given by $(t_1, t_2) \mapsto t_1$. Let $\mathcal{F} = \mathcal{O}_X$, and $H^0(X, \mathcal{O}_X) = k[t_2, t_2]$. Let $s = (t)$ be the origin in $\mathbb{A}^1_k$. Then we have that $H^0(X, \mathcal{O}_X) \otimes_A k(s) = k[t_2]$, $X_s = \mathbb{A}^1_k \setminus \{0\}$, so $H^0(X_s, \mathcal{O}_{X_s}) = k[t_2, t_2^{-1}]$.

We approach this question with out good friend Čech cohomology. Let $(U_i)_{0 \le i \le N}$ be an affine open cover of X, so $U_i = \operatorname{Spec} R_i$, and $U_I = \bigcap_{i \in I} U_i = \operatorname{Spec} R_I$ is also affine for $\emptyset \neq I \subset \{0, \dots, N\}$, because X is separated. Then we have that $X_B = \bigcup U_i'$, where $U_I' = U_I \times_A \operatorname{Spec} B = \operatorname{Spec}(R_I \otimes_A B)$, and $\mathcal{F}_B(U_I') = F(U_I) \otimes_A B$. We have that

$$\check{\mathcal{C}}^\bullet((U_i'), \mathcal{F}_B) = \check{\mathcal{C}}^\bullet((U_i), \mathcal{F}) \otimes_A B \tag{5.71}$$

and

$$H^*(X_B, \mathcal{F}_B) = H^*(\check{\mathcal{C}}((U_i), \mathcal{F}) \otimes_A B). \tag{5.72}$$

By homological algebra, we have a map

$$H^*(\check{\mathcal{C}}((U_i), \mathcal{F})) \otimes_A B \to H^*(\check{\mathcal{C}}((U_i), \mathcal{F}) \otimes_A B), \tag{5.73}$$

so we get a map

$$\beta_* : H^*(\check{\mathcal{C}}((U_i), \mathcal{F})) \otimes_A B \to H^*(X_B, \mathcal{F}_B) \tag{5.74}$$

Here is one important case where β_p is always an isomorphism.

Theorem 5.26 (Flat base change)**.** *If B is a* flat *A-algebra, then $H^*(X, \mathcal{F}) \otimes_A B \to H^*(X_B, Y_B)$ is an isomorphism for any X, and any quasicoherent sheaf $\mathcal{F}$.*

Proof. Since B is flat over A, β_* is an isomorphism by Proposition 4.2. □

Example 5.27. If $X \to \operatorname{Spec} k$, k a field, then for any field extension K/k, $H^*(X_k, \mathcal{F}_k) = H^*(X, \mathcal{F}) \otimes_k K$.

Example 5.28. If $X = A_\mathfrak{p}$, $\mathfrak{p} \in \operatorname{Spec} A$, then $H^*(X, \mathcal{F})_\mathfrak{p} = H^*(X \times_A A_\mathfrak{p}, \mathcal{F}_{A_\mathfrak{p}})$. Thus we can often reduce understanding cohomology to the case of a local base.

This is one of the great things about scheme theory: you can localize on the base, so instead of studying $X \to \operatorname{Spec} A$, you instead study $X_\mathfrak{p} \to \operatorname{Spec} A_\mathfrak{p}$.

Importantly, we also need to study the non-closed points. There were attempts to simplify scheme theory by eliminating non-closed points, but this is a stupid idea.

In the general case, we will replace $\check{\mathcal{C}}^\bullet((U_i, \mathcal{F})$ by some smaller complex which also computes cohomology.

Assume that $f : X \to Y = \operatorname{Spec} A$ is proper, $\mathcal{F}$ is coherent, and flat over A. Then $H^p(X, \mathcal{F})$ is then a finite A-module, which is zero for p sufficiently large. For example, if $X = \bigcup_{0 \le i \le N} U_i$, then $\check{\mathcal{C}}^p = 0$ for $p > N$.

Theorem 5.29 (On which everything rests)**.** *Let $f : X \to \operatorname{Spec} A$ be proper, A Noetherian, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module, which is A-flat. Assume that $H^p(X, \mathcal{F}) = 0$ for all $p > n$. Then there exists a complex*

$$L^0 \to L^1 \to \cdots \to L^n \to 0 \to 0 \to \cdots \tag{5.75}$$

of finite flat A-modules L^p such that for all A-algebras B, there is an isomorphism

$$H^*(L^\bullet \otimes_A B) \to H^*(X_B, \mathcal{F}_B) \tag{5.76}$$

which is functorial in B.

Remark 5.30.
1. The above theorem tells us there is a complex $L^\bullet$ which computes not just the cohomology of X, but the cohomology of all base changes.
2. In fact, for any A-module M, $H^*(L^\bullet \otimes_A M) = H^*(X, \mathcal{F} \otimes_A M) = H^*(X, \mathcal{F}_{\mathcal{O}_X} f^{-1}\tilde{M})$.
3. Moreover, we can arrange that the L^ps are finite and free for all $p \neq 0$.
4. Since A is Noetherian, L is a finite flat A-module if and only if L is finite locally free if and only if L is finite and projective.

To prove this, we use the following result from homological algebra.

Theorem 5.31. *Let A be Noetherian, and let $K^\bullet$ be a complex of A-modules, and assume that $H^p(K^\bullet) = 0$ for all $p > m$ and $H^p(K^\bullet)$ is finite for all p. Then*

(i) There exists a complex

$$L^0 \to L^1 \to \cdots \to L^n \to 0 \to 0 \to \cdots \tag{5.77}$$

of finite A-modules with $L^0, \dots, L^m$ free, and a morphism $f : L^\bullet \to K^\bullet$ of complex such that $H^*(f) : H^*(L^\bullet) \to H^*(K^\bullet)$ is an isomorphism.

(ii) Suppose $p \ge 0$, K^p is A-flat, and zero for p sufficiently large. Then L^0 is flat and for every A-module M,

$$H^*(f \otimes \mathrm{id}_M) : H^*(L^\bullet \otimes_A M) \to H^*(K^\bullet \otimes_A M) \tag{5.78}$$

is an isomorphism.

We rely on the following lemma.

Lemma 5.32. *Let*

$$0 \to C^0 \to \cdots \to C^N \to 0 \tag{5.79}$$

be an exact sequence of flat A-modules. For for all A-modules M,

$$0 \to C^0 \otimes M \to \cdots \to C^N \otimes M \to 0 \tag{5.80}$$

is also exact.

Proof. Proceed by induction, using Question 4 on Sheet 2. □

Now we can prove our homological algebra theorem.

Proof of Theorem 5.31. (i) We start by defining L^m and working backwards. Choose L^m free, finite with $\overline{f^m} : L^m \twoheadrightarrow H^m(K^\bullet)$. As L^m is free, we can lift $\overline{f^m}$ to a map $f^m : L^m \to \ker d_K^m$. Note that this map might not be surjective.

Now for the inductive step. Assume the for some $n \le m$ we have a commuting diagram

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & L^n & \xrightarrow{d_L^n} & L^{n+1} & \longrightarrow & \cdots & \longrightarrow & L^m & \xrightarrow{d_L^m} & 0 \\
\downarrow & & & & \downarrow & & \downarrow{\scriptstyle f^n} & & & & & & \downarrow{\scriptstyle f^m} & & \\
K^0 & \longrightarrow & \cdots & \longrightarrow & K^{n-1} & \longrightarrow & K^n & \longrightarrow & K^{n+1} & \longrightarrow & \cdots & \longrightarrow & K^m & \longrightarrow & K^{m+1}
\end{array}$$

Such that L^p are finite and free, and $H^p(f^\bullet) : H^p(L^\bullet) \to H^p(K^\bullet)$ is an isomorphism for $n < p \le m$, and $\ker d_L^n \twoheadrightarrow H^n(K^\bullet)$ is surjective.

Our base case is when $n = m$, and we have that $\ker d_L^m = L^m$, so $\ker d_L^m \twoheadrightarrow H^m(K^\bullet)$ is surjective because $\overline{f^m}$ is surjective.

Choose a finite free A-module P with an exact sequence

$$P \xrightarrow{g} \ker d_L^n \longrightarrow H^n(K) \longrightarrow 0,$$

so that $\operatorname{coker} g = H^n(K)$. We then have that $f^n \circ g(P) \subset \operatorname{im} d_K^{n-1} \subset K^n$. Another way to see this is that as before, since P is free, we can lift g to $\tilde{g} : P \to K^{n-1}$ such that $d_K^{n-1} \circ \tilde{g} = f^n \circ g$.

We want a surjection, so we choose a finite free Q such that we have a map $h : Q \to \ker d_K^{n-1}$ such that $Q \twoheadrightarrow H^{n-1}(K^\bullet)$ is surjective. Then we get a commuting diagram

$$\begin{array}{ccccc}
Q \oplus P & \xrightarrow{0 \oplus g} & L^n & \longrightarrow & L^{n+1} \\
\downarrow{\scriptstyle h \oplus \tilde{g}} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} \\
K^{n-1} & \longrightarrow & K^n & \longrightarrow & K^{n+1}
\end{array}$$

and if we set $L^{n-1} = Q \oplus P$, $d_L^{n-1} = 0 \oplus g$, and $f^{n-1} = h \oplus \tilde{g}$, then we have that $H^n(f^{n-1})$ is an isomorphism and $\ker d_L^{n-1} \twoheadrightarrow H^{n-1}(K^\bullet)$ is surjective.

At $n = 0$, we get a finite free module $\varphi : L^0 \twoheadrightarrow H^0(K^\bullet) = \ker d_K^0$. We replace L^0 by $L^0 / \ker \varphi$, which is finite but not necessarily free.

(ii) Let $C^\bullet$ be the complex (called a "mapping cone") $C^p = L^p \oplus K^{p-1}$ with differential

$$d_C : (x, y) \mapsto (dx, f(x) - dy). \tag{5.81}$$

This is a complex as

$$d^2(x, y) = d(dx, fx - dy) = (d^2x, fdx - dfx + d^2y) = 0. \tag{5.82}$$

We have a short exact sequence of complexes

$$0 \to (K^{\bullet-1}, -d_K) \to C^\bullet \to L^\bullet \to 0 \tag{5.83}$$

giving a long exact sequence of cohomology

$$H^{n-1}(K^\bullet) \to H^n(C^\bullet) \to H^n(L^\bullet) \to H^n(K^\bullet) \tag{5.84}$$

and we see that the transition map (the last arrow) is $H^n(f)$, which is an isomorphism. So

$$0 \to C^0 \to C^1 \to \cdots \to C^n \to 0 \tag{5.85}$$

is exact and $C^1, \cdots C^N$ are all flat, so $L^0 = C^0$ is flat by Question 4 on Sheet 2, and (5.83) remains exact after tensoring by M, again by Question 4 Sheet 2. So we get a long exact sequence on cohomology

$$H^n(C^\bullet \otimes M) \to H^n(L^\bullet \otimes M) \to H^n(K \otimes M) \tag{5.86}$$

and the last arrow, which is $H^n(f \otimes \mathrm{id})$, is an isomorphism by Lemma 5.32. □

Now we prove the big theorem.

Proof of Theorem 5.29. We know that $H^*(X_B, \mathcal{F}_B) = H^*(\check{\mathcal{C}}^\bullet((U_i), \mathcal{F}) \otimes_A B)$ if (U_i) is a finite open affine cover of X. Applying Theorem 5.31 gives the result. □

We give some consequences of Theorem 5.29. First we look at H^0. We have that

$$H^0(X_B, \mathcal{F}_B) = \ker(d_L^0 \otimes \mathrm{id}_B : L^0 \otimes B \to L^1 \otimes B) \tag{5.87}$$

and for any finite flat (in other words, finite and locally free) A-module M,

$$M \otimes_A B = \mathrm{Hom}_A(M^\vee, A) \otimes_A B = \mathrm{Hom}_A(M^\vee, B) \tag{5.88}$$

where $M^\vee = \mathrm{Hom}_A(M, A)$. Letting $Q = \mathrm{coker}((d_L^0)^\vee : (L^1)^\vee \to (L^0)^\vee)$, we get the following corollary.

Corollary 5.33. *There exists a finite A-module Q such that for all B,*

$$H^0(X_B, \mathcal{F}_B) \cong \mathrm{Hom}_A(Q, B) \tag{5.89}$$

and this isomorphism is functorial in B. So the functor $B \to H^0(X_B, \mathcal{F}_B)$ is represented by Q.

Remark 5.34. The fact that $L^\bullet$ is finite and flat is important. We already have a complex which computes H^*, namely the Čech complex, but it is huge.

Remark 5.35. Let $\mathcal{F}(s)$ be the pullback of $\mathcal{F}$ to $X_s = f^{-1}(s) = X \times_A k(s)$. So we have $\mathcal{F}(s) = \pi^*\mathcal{F}$, where $\pi : X \times k(s) \to X$. Then $\mathcal{F}(s)$ is a coherent $\mathcal{O}_{X_s}$-module.

Remark 5.36 (Alternative definition for Hilbert polynomials)**.** Let $M_\bullet$ be a graded finite module for $k[T_0, \ldots, T_n] = R$. Then $\dim M_n$ is a polynomials for n sufficiently large.

Now, let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^N_k$. Then we can set

$$M_\bullet = \bigoplus_n H^0(\mathbb{P}^N, \mathcal{F}(n)) \tag{5.90}$$

which is a graded R-module, where

$$R = \bigoplus_n H^0(\mathbb{P}^n, \mathcal{O}(n)). \tag{5.91}$$

For n sufficiently large, $H^p(\mathbb{P}^N, \mathcal{F}(n)) = 0$ for all $p > 0$. So $\chi(\mathbb{P}^N, \mathcal{F}(n)) = \dim H^0(\mathcal{F}(n))$ for n sufficiently large, as $\mathcal{O}_{\mathbb{P}}(1)$ is ample.

Corollary 5.37 (Semicontinuity for H^0)**.** *For every $r \geq 0$, the set $Z_r = \{s \in \operatorname{Spec} A \mid \dim_{k(s)} H^0(X_s, \mathcal{F}(s)) \geq r\}$ is closed. Thus the function $A \to \mathbb{Z}$, $s \mapsto \dim_{k(s)} H^0(X_s, \mathcal{F}(s))$ is semicontinuous.*

Proof. Locally around $s \in \operatorname{Spec} A$, we have that $L^0 \cong A^m$ and $L^1 \cong A^1$ are free, and

$$H^0(X_s, \mathcal{F}(s)) = \ker(d^0_L \otimes \mathrm{id} : L^0 \otimes k(s) \to L^1 \otimes k(s)) \tag{5.92}$$

so its dimension is the rank of the matrix of $d^0_L \otimes \mathrm{id}$, and so Z_r is the set of s where the nullity of the matrix is greater than r, so it is the set of s where all $(m - r + 1)$-minors of the matrix vanish in $k(s)$. This is a closed set.

Another way to prove this which might be more correct: We have that $H^0(X_s, \mathcal{F}(s)) = \operatorname{Hom}_A(Q, k(s))$ for some finite A-module Q. Taking tensor products gives $H^0(X_s, \mathcal{F}(s)) = \operatorname{Hom}_{k(s)}(Q \otimes k(s), k(s))$, so

$$\dim_{k(s)} H^0(X_s, \mathcal{F}(s)) = \dim_{k(s)}(Q \otimes k(s)) \tag{5.93}$$

so by Proposition 3.31, Z_r is closed. □

We will show that this holds for all H^p on the next example sheet.

Corollary 5.38. *Assume* $\operatorname{Spec} A$ *is connected and let f be proper, $\mathcal{F}$ flat over A. Then $\chi(X, \mathcal{F}(s))$ is independent of s.*

If $X \subset \mathbb{P}^N_A$ is projective, then the Hilbert polynomial $P(X_s, \mathcal{F}(s), t)$ is independent of s.

Proof. The first statement implies the second by the definition of the Hilbert polynomial. For the first statement, covering $\operatorname{Spec} A$ by small open affines, we may assume that all of the L^i are free. Then

$$\begin{aligned}
\chi(X_s, \mathcal{F}(s)) &= \sum_{p \geq 0} (-1)^p \dim H^p(X_s, \mathcal{F}(s)) \\
&= \sum_{p \geq 0} (-1)^p \dim_{k(s)} H^p(L^\bullet \otimes_A k(s)) \\
&= \sum_{p \geq 0} (-1)^p \dim_{k(s)} L^p \otimes_A k(s) \\
&= \sum_{p \geq 0} \operatorname{rank}_A L^p
\end{aligned} \tag{5.94}$$

as the L^p are all free. $\square$

At the top, in the original scheme X, we have that if $H^p(X, \mathcal{F}) = 0$ for all $p > n$, then for $p > n$, $L^p = 0$, so $H^p(X_B, \mathcal{F}_B) = 0$ for all $p > n$. In particular, $H^p(X_s, \mathcal{F}(s)) = 0$ for all s, and all $p > n$. Then

$$\begin{aligned} H^n(X_B, \mathcal{F}_B) &= \operatorname{coker}(L^{n-1} \otimes_A B \to L^n \otimes_A B) \\ &= \operatorname{coker}(L^{n-1} \to L^n) \otimes_A B \\ &= H^n(X, \mathcal{F}) \otimes_A B, \end{aligned} \tag{5.95}$$

so our map $\beta_n : H^n(X, \mathcal{F}) \otimes_A B \to H^n(X_B, \mathcal{F}_B)$ is an isomorphism. We use that the tensor product is right exact in the calculation above.

Now, suppose that A is reduced. Then

$$A \to \prod_{\mathfrak{p} \in \operatorname{Spec} A} A_{\mathfrak{p}}/\mathfrak{p} = \prod_{s \in \operatorname{Spec} A} k(s) \tag{5.96}$$

is injective, as the kernel is $\bigcap \mathfrak{p} = \sqrt{0} = 0$. As $H^n(X, \mathcal{F})$ is a finite A-module, $H^n(X, \mathcal{F}) = 0$ if and only if for all $s \in \operatorname{Spec} A$, $H^n(X, \mathcal{F}) \otimes k(s) = 0$ if and only if for all $s \in \operatorname{Spec} A$, $H^n(X_s, \mathcal{F}(s)) = 0$. If these conditions hold, we can replace n by $n-1$. Continuing this until we get two nonzero cohomologies, we have the following corollary.

Corollary 5.39. *Assume that A is reduced, and let $p \geq 0$. The following are equivalent:*

(i) For all $i \geq p$, and for all $s \in \operatorname{Spec} A$, $H^i(X_s, \mathcal{F}(s)) = 0$.

(ii) For all $i \geq p$, $H^i(X, \mathcal{F}) = 0$.

6 Group Schemes

Fix a field k. Let Sch/k be the category of k-schemes and k-morphisms, and let Aff/k be the category of *affine* k-schemes, so $\operatorname{Spec} A$ for A a k-algebra.

Let X, S be k-schemes. Then we write

$$X(S) = \operatorname{Mor}_{\operatorname{Sch}/k}(S, X) \tag{6.1}$$

to be the S-valued points of X. If $S = \operatorname{Spec} R \in \operatorname{Aff}/k$, then we write $X(R) = X(S)$, for the set of R-valued points.

Example 6.1. If $X = V(I) \subset A_k^N$ is the vanishing set of some set of polynomials, then

$$X(R) = \{(x_1, \ldots, x_N) \in R^N \mid \forall f \in I, f(x_1, \ldots, x_N) = 0\}. \tag{6.2}$$

This is because a morphism $\operatorname{Spec} R \to X$ is determined by a ring map $k[t_1, \ldots, t_N]/I \to R$, which is just a map $k[t_1, \ldots, t_N] \to R$ which is zero on I, and these maps are determined by their values on $t_i \mapsto x_i$, as the image of k is fixed since we are working in the category of k-schemes.

Remark 6.2. If $k = \overline{k}$ is algebraically closed, then $X(k)$ is in canonical bijection with the closed points of k.

Definition 6.3. A k-group scheme is a scheme $G \in \mathrm{Sch}/k$ together with a k-morphism

$$m : G \times_k G \to G \tag{6.3}$$

such that for all k-algebras R, the induced map

$$m_R : G(R) \times G(R) \to G(R) \tag{6.4}$$

which is defined by $m_R(x_1, x_2)(r) = m(x_1(r), x_2(r))$ makes $G(R)$ into a group.

Example 6.4.

1. The additive group: Let $\mathbb{G}_a = \operatorname{Spec} k[t] \cong \mathbb{A}^1_k$. Then we have

$$\begin{aligned} m^{\#} : k[t] &\to k[t_1, t_2] \\ t &\mapsto t_1 + t_2 \end{aligned} \tag{6.5}$$

   We then have that $\mathbb{G}_a(R) = (R, +)$.

2. The multiplicative group: Let $\mathbb{G}_m = \operatorname{Spec} k[t, 1/t] = \mathbb{A}^1_k \setminus \{0\}$. Then we have

$$\begin{aligned} m^{\#} : k[t, 1/t] &\to k[t_1, t_2, 1/(t_1 t_2)] \\ t &\mapsto t_1 t_2 \end{aligned} \tag{6.6}$$

   and then $\mathbb{G}_m(R) = (R^\times, \times)$.

3. The general linear group: Let

$$\mathrm{GL}_n = \operatorname{Spec} k[\{t_{ij}\}_{1 \le i,j \le n}, 1/\det(t_{ij})] \tag{6.7}$$

   Then $\mathrm{GL}_n(R)$ is the group of invertible $n \times n$ matrices, as adjoining $1/\det(t_{ij})$ to $\operatorname{Spec} k[\{t_{ij}\}_{1 \le i,j \le n}]$ is like ensuring that $\det(a_{ij}) \neq 0$. We have that $m^{\#}$ is just matrix multiplication.

We can more generally work over any base scheme B instead of $\operatorname{Spec} k$.

Definition 6.5. A group scheme over B is a scheme G/B, and a map $m : G \times_B G \to G$ such that for all affine $\operatorname{Spec} R \to B$, $(G(R), m_R)$ is a group.

Typically, the "topological" points of G don't form a group, but the closed points might.

Example 6.6. If we consider $\mathbb{G}_a = \mathbb{A}^1_k$, we can't make the points of this into a group in a sensible way, as the nonclosed point is hard to deal with.

Also, the closed points with residue field not equal to k give problems.

Remark 6.7. For a general scheme X, the points of the underlying space don't determine X. But the R-valued points do.

Let's work over k, and let $X \in (\mathrm{Sch/k})$, and for every k-scheme we have $X(S) = \mathrm{Mor}_k(S, X)$. If $g : S' \to S$ is a morphism, we get by composition a map

$$\begin{aligned} X(S) &\to X(S') \\ f &\mapsto f \circ g \end{aligned} \tag{6.8}$$

which is compatible with $S'' \to S' \to S$. Thus we get a contravariant functor $h_X : (\mathrm{Sch}/k) \to \mathrm{Set}$, sending $S \mapsto X(S)$ and $g \mapsto (\_\_ \circ g)$. This also gives a covariant functor from the category of k-algebras to sets, sending $R \mapsto X(R) = X(\operatorname{Spec} R)$. We then have the Yoneda lemma.

Lemma 6.8 (Yoneda)**.** *The set of k-morphisms from X to Y is in bijection with the set of natural transformations $h_X \to h_Y$.*

Proof. Trivial, did this in alggeo. □

In fact, we have a stronger result for schemes that allows us to consider only affines.

Lemma 6.9 (Yoneda for schemes)**.** *Let X, Y be k-schemes, and let h'_X be the restriction of h_X to the category of affine schemes (so send $S = \operatorname{Spec} R \mapsto X(S)$). Then the set of morphisms $\operatorname{Mor}_k(X, Y)$ is in bijection with the set of natural transformations $h'_X \to h'_Y$.*

Proof. Given a natural transformation from $h'_X \to h'_Y$, we want to construct a morphism $X \to Y$. A natural transformation $h'_X \to h'_Y$ is a collection of morphisms $f_S : X(S) \to Y(S)$ for each $S = \operatorname{Spec} R$ affine which are compatible with the functors h'_X, h'_Y.

Cover X by open affines $U \in \mathcal{U}$, and let $j_U : U \hookrightarrow X$ be the inclusion. Then $f_U(j_U) \in Y(U)$. For any two open affines $U, U' \subset X$, let $V \subset U \cap U'$ be an open affine in the intersection. Let $i : V \hookrightarrow U$ and $i' : V \hookrightarrow U'$ be the two inclusions. We have that $h_X(i)(j_U) = h_X(i')(j_{U'}) = j_V$, since we are just composing inclusions. Since f is a natural transformation, the following diagram commutes:

$$\begin{array}{ccc} X(U) & \xrightarrow{f_U} & Y(U) \\ \downarrow{\scriptstyle h_X(i)} & & \downarrow{\scriptstyle h_Y(i)} \\ X(V) & \xrightarrow{f_V} & Y(V) \\ {\scriptstyle h_X(i')}\uparrow & & {\scriptstyle h_Y(i')}\uparrow \\ X(U') & \xrightarrow{f_{U'}} & Y(U') \end{array}$$

Thus we have that $h_Y(i') \circ f_{U'}(j_{U'}) = f_V(j_V)$

$$\begin{aligned} f_{U'}(j_{U'})|_V &= f_{U'}(j_{U'}) \circ i' \\ &= h_Y(i') \circ f_{U'}(j_{U'}) \\ &= f_V(j_V) \\ &= h_Y(i) \circ f_U(j_U) \\ &= f_U(j_U)|_V. \end{aligned} \tag{6.9}$$

Thus $f_{U'}(j_{U'})$ and $f_U(j_U)$ agree on V, so we can glue them together to get a morphism $f : X \to Y$.

We can check that the map from $\operatorname{Mor}_k(X, Y)$ to $\operatorname{NatTrans}(h_X, h_Y)$ given as before is the inverse. □

Here are some more nice consequences of the Yoneda lemma. I'm actually not sure how some of these follows from Yoneda so I've done them by brute force.

Proposition 6.10. *Suppose G is a k-group scheme. Then*

(i) For all $S \in (\operatorname{Sch}/k)$, $G(S)$ is a group.

(ii) For all morphisms $S' \to S$, $G(S) \to G(S')$ is a group homomorphism.

Proof. (i) Let $S \in (\mathrm{Sch}/k)$ and let $\mathcal{U}$ be a cover of S by affines U. Any $x \in G(S)$ is determined by its restriction to affines. This defines an injection

$$\begin{aligned} G(S) &\hookrightarrow \prod_{U \in \mathcal{U}} G(U) \\ x &\mapsto (x|_U)_U. \end{aligned} \tag{6.10}$$

We want to show that the image of $G(S)$ under this map is a subgroup of $\prod G(U)$, which is a group itself, under the operation

$$(x \cdot y)(s) = m_S(x, y)(s) = m(x(s), y(s)) \tag{6.11}$$

For any $s \in S$, we have the $s \in U$ for some $U \in \mathcal{U}$, so

$$\begin{aligned} (x \cdot y)(s) &= m(x(s), y(s)) \\ &= m(x|_U(s), y|_U(s)) \\ &= m_U(x|_U, y|_U)(s) \end{aligned} \tag{6.12}$$

so we have that

$$m_S(x, y) \mapsto (m_U(x|_U, y|_U))_U \tag{6.13}$$

under the map (6.10). Thus $G(S)$ is clearly a subgroup, hence a group.

(ii) Let $f : S' \to S$ be a morphism of k-schemes. Then we have a map $G(S) \to G(S')$ sending $x \mapsto x \circ f$. We want to show that this is a homomorphism, and for all $p \in S'$ we have that

$$\begin{aligned} (m_S(x, y) \circ f)(p) &= m(x \circ f(p), y \circ f(p)) \\ &= m_{S'}(x \circ f, y \circ f)(p) \end{aligned} \tag{6.14}$$

which shows that we have a group homomorphism.

(ii) Alternative: Let $g : S' \to S$. The fact that m is a natural transformation immediately gives $xy \circ g = (x \circ g)(y \circ g)$. □

Corollary 6.11. *There exists $e \in G(k)$, and a morphism $i : G \to G$ such that for all S, $e \mapsto e_S$ under the map $G(k) \to G(S)$ induced by $S \to k$, and $i_S : G(S) \to G(S)$ is the inverse.*

Proof. Let e be the identity in $G(k)$. Then since $G(k) \to G(S)$ is a group homomorphism, $e \mapsto e_S$.

We have that $G(G) = \mathrm{Mor}(G, G)$ is a group, and let i be the inverse of the identity morphism in $G(G)$. Then we have that $m_G(\mathrm{id}_G, i) = e_G$. Thus, $i_S : x \to i \circ x$ is a map from $G(S) \to G(S)$, we have that

$$\begin{aligned} m_S(x, i_S(x))(p) &= m(\mathrm{id} \circ x(p), i \circ x(p) \\ &= m_G(\mathrm{id}, i)(x(p)) \\ &= e_G(x(p)) \\ &= x^{\#}(e_G) \\ &= e_S \end{aligned} \tag{6.15}$$

where $x^{\#}$ is the group homomorphism $G(G) \to G(S)$ sending $g \to g \circ x$.

Alternative proof of inverse: For each S, we have $i_S : G(S) \to G(S)$ sending $x \mapsto x^{-1}$. These maps are compatible: if $f : S' \to S$, we want to show the diagram below commutes:

$$
\begin{array}{ccc}
G(S) & \xrightarrow{i_S} & G(S) \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\
G(S') & \xrightarrow{i'_S} & G(S').
\end{array}
$$

Since f^* is a group homomorphism, we have that

$$
\begin{aligned}
(i_S(x) \circ f)(x \circ f) &= (i_S(x)x) \circ f \\
&= e_S \circ f \\
&= e_{S'}.
\end{aligned} \tag{6.16}
$$

Thus the compatible family of maps i_S gives rise to a natural transformation $i : G \to G$ by Yoneda stuff. $\square$

Definition 6.12 (Alternative definition of group schemes)**.** A k-group scheme can equivalently be defined as

$$
(G, m : G \times G \to G, e \in G(k), i : G \to G) \tag{6.17}
$$

satisfying suitable axioms:

1. (Associativity) The diagram

$$
\begin{array}{ccccc}
(G \times_k G) \times_k G & \xrightarrow{m \times \mathrm{id}} & G \times G & & \\
\downarrow{\scriptstyle \cong} & & & \searrow{\scriptstyle m} & \\
G \times_k (G \times_k G) & \xrightarrow{\mathrm{id} \times m} & G \times G & \xrightarrow{m} & G
\end{array}
$$

and others which I will write down later.

Another alternative definition is that a group scheme is a group object in (Sch/k).

Another alternative definition is that a group scheme is a representable contravariant functor $(\mathrm{Sch}/k) \to \mathrm{Grp}$.

Definition 6.13. A *homomorphism of group schemes* is a morphism $f : G \to G'$ such that for all k-algebras R, $f_R : G(R) \to G'(R)$ is a homomorphism where f_R is the obvious map sending $x \mapsto f \circ x$.

Equivalently, it is a morphism such that $f_S : G(S) \to G'(S)$ is a homomorphism for all $S \in (\mathrm{Sch}/k)$.

Equivalently, it is a morphism making the following diagram commute:

$$
\begin{array}{ccc}
G \times G & \xrightarrow{f \times f} & G' \times G' \\
\downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m'} \\
G & \xrightarrow{f} & G'
\end{array}
$$

Definition 6.14. A *closed subgroup scheme* of a k-group scheme G is a closed subscheme $i : H \hookrightarrow G$ such that for all R, $H(R) \subset G(R)$ is a subgroup. If so, H is a group scheme, and i is a homomorphism of group schemes.

Proof. By the Yoneda lemma, for every $S \in (\mathrm{Sch}/k)$, the morphism m_S factors through

$$\begin{array}{ccc} (H \times H)(S) & \xrightarrow{(i \times i)(S)} & (G \times G)(S) \\ \downarrow & & \downarrow m_S \\ H(S) & \xrightarrow{i(S)} & G(S) \end{array}$$

since it does for S affine. Thus by the Yoneda lemma, we have that m factors through

$$\begin{array}{ccc} H \times H & \xrightarrow{i \times i} & G \times G \\ \downarrow & & \downarrow m \\ H & \xrightarrow{i} & G \end{array}$$

which gives H the structure of a group scheme. ☐

Example 6.15. 1. $e : \operatorname{Spec} k \hookrightarrow G$ is a closed subgroup scheme.

2. *Kernels:* Let $f : G \to G'$ be any homomorphism, and define $\ker f$ to be the fiber product

$$\begin{array}{ccc} \ker f & \longrightarrow & G \\ \downarrow & & \downarrow f \\ \operatorname{Spec} k & \xrightarrow{e'} & G' \end{array}$$

so $\ker f = f^{-1}(e')$, the fiber of e'.

By the definition of fiber product, $\ker f(s) = \ker(f_S : G(S) \to G'(S))$ and e' is a closed immersion, so its pullback $\ker f$ is a closed subscheme, so its a closed subgroup scheme.

3. For all R we have $\det : \mathrm{GL}_n(R) \to R^\times = \mathbb{G}_m(R)$ which is functorial in R. So by the Yoneda lemma, we have a homomorphism of group schemes

$$\det : \mathrm{GL}_n \to \mathbb{G}_m \tag{6.18}$$

with kernel $\mathrm{SL}_n = \ker(\det) = \operatorname{Spec} k[\{t_{ij}\}]/(\det(t_{ij}) - 1)$.

Remark 6.16. Quotients of group schemes are harder to make sense of.

Definition 6.17. Let $x \in G(k)$. Then the left *translation* morphism $L_x = T_x : G \to G$ is the composite

$$G = \operatorname{Spec} k \times_k G \xrightarrow{x \times \mathrm{id}_G} G \times G \xrightarrow{m} G$$

and is the unique morphism such that for all $g \in G(S)$, $T_x(g) = m(x, g) = xg$. Obviously, $T_e = \mathrm{id}_G$, and $T_{xy} = T_x \circ T_y$. So T_x is a k-automorphism of G.

More generally, let S be a k-scheme, and $x \in G(S)$. Then we can define $T_x : G \times S \to G$ as the composite

$$G \times S \xrightarrow{\mathrm{id} \times x} G \times G \xrightarrow{{}^{\mathrm{t}}m} G$$

where ${}^{\mathrm{t}}m : G \times G \to G$ is given on points by $(g, h) \mapsto m(h, g) = hg$. If $S = \operatorname{Spec} k$ this is the same as before.

The product

$$\begin{aligned} T_{x/S} : G \times S &\to G \times S \\ (g, s) &\mapsto (T_x(g, s), s) \end{aligned} \tag{6.19}$$

satisfies $T_{e/S} = \mathrm{id}_{G \times_k S}$, and $T_{x/S} \circ T_{y/S} = T_{xy/S}$, so T_x is an S-automorphism of $G \times_k S$.

Similarly, we can define right translation and check that it is the same as left translation if G is commutative.

That being said, lets define commutativity.

Definition 6.18. G is *commutative* if $G(R)$ is commutative for all R, if and only if ${}^{\mathrm{t}}m = m$ by Yoneda.

6.1 Abelian varieties

If G is a k-group scheme and k'/k any extension, then $G \times_k \operatorname{Spec} k' = G_{k'}$ is a k'-group scheme, with multiplication given by

$$m_{k'} : G_{k'} \times_{k'} G_{k'} = (G \times_k G) \times_k k' \xrightarrow{m \times \mathrm{id}} G \times_k k' \tag{6.20}$$

Recall that in this course, a k-variety is a separated k-scheme of finite type which is geometrically integral.

Definition 6.19. A k-group variety (or algebraic group, but "algebraic group" has many definitions) is a k-group scheme which is a k-variety.

Definition 6.20. An *abelian variety* (AV) over k is a k-group variety which is proper over k.

We will at some point see that all abelian varieties are projective.

Example 6.21.

1. $\mathbb{G}_a, \mathbb{G}_m, \mathrm{GL}_n$ are group varieties.
2. The simplest nontrivial abelian variety is an elliptic curve over k.

The product of two group varieties (respectively abelian varieties) is one as well. This is because if $(G, m), (G', m')$ are two k-group schemes, then so is $(G \times G', m \times m')$

$$m \times m' : (G \times G') \times (G \times G') \cong (G \times G) \times (G' \times G') \to G \times G'. \tag{6.21}$$

We also have that the product of two varieties is a variety, and the product of two proper morphisms is proper.

In particular, we can construct abelian varieties of arbitrary dimension by taking products of elliptic curves.

Remark 6.22. Some people define an algebraic group as a k-group scheme of finite type.

Proposition 6.23. *Let G be a group variety over k. Then G is smooth over k.*

Proof. It is enough to check that $G_{\overline{k}}$ is smooth over $\overline{k}$, so we may assume that $k = \overline{k}$ is algebraically closed. Then the closed points are the same as k-points. The set of smooth points is non-empty and open, so it contains a closed point, as G is of finite type over k. If $x \in G(k)$, then $T_x : G \to G$ is a k-automorphism taking e to x. So G is smooth at x if and only if it is smooth at e. So G is smooth at every closed point, so G is smooth. □

Corollary 6.24. *$\Omega_{G/k}$ is locally free if G is a k-group variety.*

In fact, $\Omega_{G/k}$ is free for any k-group scheme.

6.2 Mumford's rigidity lemma

We will discuss Mumford's rigidity lemma.

Theorem 6.25 (Mumford's Rigidity Lemma)**.** *Let X, Y, Z be k-varieties with X proper, $X(k) \neq \emptyset$. Let $f : X \times_k Y \to Z$ be a k-morphism. Suppose that for some $y_0 \in Y$, $z_0 \in Z$, we have that $X \times \{y_0\} = X \times \operatorname{Spec} k(y_0)$ is contained in $f^{-1}(z_0)$ set theoretically, so f collapses $X \times \{y_0\}$ to a point. Then there exists $g : Y \to Z$ such that $f = g \circ \mathrm{pr}_2$.*

In diagram terms we have

$$\begin{array}{ccc} X \times Y & \xrightarrow{\ f\ } & Z \\ \Big\downarrow{\scriptstyle \mathrm{pr}_2} & \overset{\exists g}{\nearrow} & \\ Y & & \end{array}$$

In words, the theorem says that given a family of morphisms $\{f_i\} : X \to Z$ with X proper, if one member of the family is constant, then so is every member.

As a reality check, let's see that we need X to be proper. Let $X = Y = Z = \mathbb{A}^1_k$, and $f(x, y) = xy$. Then f collapses $\mathbb{A}^1 \times \{0\}$ to a point, but no other fiber.

Before we prove this, we need a lemma.

Lemma 6.26. *Let X be a proper k-variety, Y any k-scheme. Let $\mathrm{pr}_2 : X \times Y \to Y$ be the projection map. Then*

$$(\mathrm{pr}_2)_* \mathcal{O}_{X \times_k Y} = \mathcal{O}_Y. \tag{6.22}$$

Proof. Consider the fiber product commutative diagram

$$\begin{array}{ccc} X \times_k Y & \xrightarrow{\ \mathrm{pr}_1\ } & X \\ \Big\downarrow{\scriptstyle \mathrm{pr}_2} & & \Big\downarrow{\scriptstyle a_X} \\ Y & \xrightarrow{\ a_Y\ } & \operatorname{Spec} k \end{array}$$

We have that a_Y is flat because everything over $\operatorname{Spec} k$ is flat, and we have that

$$\operatorname{pr}_1^* \mathcal{O}_X = \operatorname{pr}_1^{-1} \mathcal{O}_X \otimes_{\operatorname{pr}_1^{-1} \mathcal{O}_X} \mathcal{O}_{X \times Y} = \mathcal{O}_{X \times_k Y}. \tag{6.23}$$

By flat base change Proposition 4.2 (take an open cover of X and Y so that all sheaves of these schemes will be of the form $\tilde{M}$, we have that pr_2 is the base change of a_x, and base change commutes with H^0, and $\tilde{M}$ is determined by its global sections, which gives the following)

$$(\operatorname{pr}_2)_* \mathcal{O}_{X \times Y} = a_Y^* (a_X)_* \mathcal{O}_X. \tag{6.24}$$

Further, we have that as vector spaces,

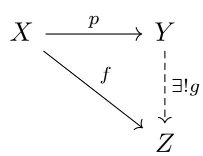
$$a_Y^* (a_X)_* \mathcal{O}_X = \mathcal{O}_Y \times_k \Gamma(X, \mathcal{O}_X) = \mathcal{O}_Y \tag{6.25}$$

since $\Gamma(X, \mathcal{O}_X) = k$ as X is a proper k-variety. □

So in the context of Theorem 6.25, we have that $(\operatorname{pr}_2)_* \mathcal{O}_{X \times_k Y} = \mathcal{O}_Y$ and as X is proper, pr_2 is a closed morphism. Thus $p = \operatorname{pr}_2$ satisfies the conditions of the following theorem, so Theorem 6.25 is a special case of the following.

Remark 6.27. The morphism $\operatorname{pr}_2 : X \times_k Y \to Y$ having a section is equivalent to $X(k) \neq \emptyset$.

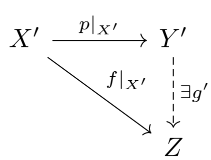
Theorem 6.28 (Rigidity)**.** *Let $p : X \to Y$ be a morphism of schemes with a section $s : Y \to X$, so $p \circ s = \mathrm{id}$. Assume p is closed, $p_* \mathcal{O}_X = \mathcal{O}_Y$, and X is integral. Suppose we have $f : X \to Z$ with Z separated. Suppose there exists points $y \in Y, z \in Z$ (not necessarily closed) such that $X_y = p^{-1}(y) \subset f^{-1}(z)$. Then $f = g \circ p$ for some unique $g : Y \to Z$. Thus we have the following diagram:*



Proof. We first do two reductions.

(1) If there exists $g : Y \to Z$ with $f = g \circ p$, then $g = g \circ (p \circ s) = f \circ s$. So $g = f \circ s$ is unique and exists if and only if $f = (f \circ s) \circ p$.

(2) Suppose there exists an open neighborhood $Y' \subset Y$ of y such that if $X' \coloneqq p^{-1}(Y')$, then $f|_{X'}$ factors through some $Y' \to Z$. So we have a diagram



Then by (1), $f|_{X'} = (f \circ s \circ p)|_{X'}$, and $X' \neq \emptyset$ as $f(X') = f \circ s \circ p(X') = f \circ s(Y') \neq \emptyset$, so X' is dense as X is irreducible. Then $f = f \circ s \circ p$ because X is reduced and Z is separated (Hartshorne exercise II.4.2).

Now, f maps $X_y = p^{-1}(y) \subset X$ to $\{z\} \subset Z$ as $X_y \subset f^{-1}(z)$ by assumption. Let $W \subset Z$ be an affine open containing z. Then $f^{-1}(W)$ is an open neighborhood of X_y. But since p is closed, and $X \setminus f^{-1}(W)$ is closed, $T = p(X \setminus f^{-1}(W)) \subset Y$ is closed and does not contain y, so taking $Y' = Y \setminus T$, we have that Y' is an open neighborhood of y and so $p^{-1}(Y') = X'$ is open. But since $p_*\mathcal{O}_X = \mathcal{O}_Y$, $\Gamma(X', \mathcal{O}_{X'}) = \Gamma(Y', \mathcal{O}_{Y'})$, so $f|_{X'}$ factors through Y'. So we are done by (2).

□

Corollary 6.29. *Let X be an abelian variety over k, and G any group variety over k. If $f : X \to G$ is any morphism, and $g = f(e) \in G(k)$, then $T_{g^{-1}} \circ f$ is a homomorphism.*

Proof. $T_{g^{-1}}$ sends $g \to e_G$, so setting $f' = T_{g^{-1} \circ f}$, it is enough to show that $f'(e_x) = e_g$ implies that f is a homomorphism. So replace f by f'.

Consider $h : X \times X \to G$, given on points by

$$(x, y) \mapsto f(x)f(y)f(xy)^{-1} \tag{6.26}$$

Then $h(X \times \{e_X\}) = h(\{e_X\} \times X) = \{e_G\} \subset G$. By rigidity, we have $X \times \{x\}$ is mapped to a point for any x, and $\{x\} \times X$ is also mapped to a point for any x, so $h(x) = e_G$ for all x. Thus $f(x)f(y)f(xy)^{-1} = e_G$ for all x, y, so $f(xy) = f(x)f(y)$. Thus f is a homomorphism. □

Corollary 6.30. *There is only one possible group structure on an abelian variety once e is distinguished.*

Proof. If X is an abelian variety and $f : X \to X$ is an isomorphism of schemes taking $e \to e$, then f is also an isomorphism of group schemes by Corollary 6.29. □

Example 6.31. We have that $\mathbb{G}_a \times \mathbb{G}_a \times \mathbb{G}_a$ and U_3 are both isomorphic to $\mathbb{A}^3$ as schemes, but they have different group structures.

Corollary 6.32. *Any abelian variety is commutative.*

Proof. Consider $i : X \to X$ the inverse map. As $i(e) = e$, by Corollary 6.29, i is a homomorphism. But then we have that for each $S \in (\mathrm{Sch}/k)$, i_S is a group homomorphism on $X(S)$, so $(xy)^{-1} = x^{-1}y^{-1}$ so $xy = yx$. □

7 Seesaw, Cube, Square: Line bundles on abelian varieties

By a *line bundle* on a scheme X, we mean an invertible $\mathcal{O}_X$-module, or equivalently a locally free $\mathcal{O}_X$-module of rank 1.

Definition 7.1. The Picard group of X, $\mathrm{Pic}(X)$ is the isomorphism classes of line bundles on X, and is a group under the tensor product $\otimes_{\mathcal{O}_X}$. We often write $\mathcal{L} \in \mathrm{Pic}(X)$ if we mean "$\mathcal{L}$ is a line bundle on X".

Theorem 7.2 (Seesaw Theorem)**.** *Let X, Y be a k-varieties, with X proper. Let $\mathcal{L}$ be a line bundle on $X \times_k Y$. Suppose for all closed $y \in Y$, $\mathcal{L}(y) = i_y^*\mathcal{L}$ is trivial, where $X \times \{y\} = \mathrm{pr}_2^{-1}(y) = X \times_k \operatorname{Spec} k(y)$ and $i_y : X \times \{y\} \hookrightarrow X \times Y$.*

Then there exists a line bundle M on Y such that $\mathcal{L} \cong \mathrm{pr}_2^* M$. Moreover, $M \cong (\mathrm{pr}_2)_*\mathcal{L}$ is unique up to isomorphism.

Proof. First we show that every $y \in Y$ has an affine neighborhood V such that $\mathcal{L}|_{X\times_k V} \cong \mathcal{O}_{X\times_k V}$. For this, we may assume $Y = \operatorname{Spec} A$ is affine. Then by Corollary 5.33 (see also Example 5.24) there exists a finite A-module Q such that $H^0(X \times \{y\}) \cong \operatorname{Hom}_A(Q, k(y))$. Since $H^0(X \times \{y\}, \mathcal{L}(y)) \cong H^0(X \times \{y\}, \mathcal{O}_{X\times\{y\}} = k \times_k k(y) = k(y)$. Now, let $y = \mathfrak{m} \in \operatorname{Spec} A$, so that $k(y) = A/\mathfrak{m}$. Then we have that

$$Hom_A(Q, A/\mathfrak{m}) = \operatorname{Hom}_{A/\mathfrak{m}}(Q \otimes_A A/\mathfrak{m}, A/\mathfrak{m}) = \operatorname{Hom}_{A/\mathfrak{m}}(Q/\mathfrak{m}Q, A/\mathfrak{m}) = A/\mathfrak{m} \tag{7.1}$$

so $\dim_{A/\mathfrak{m}}(Q/\mathfrak{m}Q) = 1$ for all maximal $\mathfrak{m} \subset A$. Thus by Proposition 3.31, Q is locally free of rank 1 (using that Y is a finitely generated k-algebra, so its closed points are dense).

So every $y \in Y$ does have an open neighborhood $V = \operatorname{Spec} B$ for which the corresponding A-module Q is free. Thus $Q = Bu$, say, for some $u \in B$. Then

$$H^0(X \times_k V, \mathcal{L}) = \operatorname{Hom}_B(Q, B) = B(u)^\vee \tag{7.2}$$

which is free generated by the dual basis. At each closed $y \in Y$, $u^\vee$ maps to a nonzero section of $H^0(X \times \{y\}, \mathcal{L}(y)) = \operatorname{Hom}(\mathcal{O}_{X\times\{y\}}, \mathcal{L}(y))$ as $\mathcal{L}(y) \cong \mathcal{O}_{X\times\{y\}}$ is trivial. Thus $u^\vee$ maps to a nowhere vanishing section of $H^0(X \times_k V, \mathcal{L})$, so it gives an isomorphism

$$u^\vee : \mathcal{O}_{X\times V} \to \mathcal{L}|_{X\times V} \tag{7.3}$$

By Lemma 6.26, $(\mathrm{pr}_2)_*(\mathcal{O}_{X\times V}) = \mathcal{O}_V$, so the adjunction

$$\mathrm{pr}_2^*(\mathrm{pr}_2)_*\mathcal{O}_{X\times V} \to \mathcal{O}_{X\times V} \tag{7.4}$$

is an isomorphism (as $\mathrm{pr}_2^* \mathcal{O}_V = \mathcal{O}_{X\times V}$ trivially), so

$$\mathrm{pr}_2^*(\mathrm{pr}_2)_*\mathcal{L}|_{X\times V} \cong \mathcal{L}|_{X\times V}. \tag{7.5}$$

For general Y, this shows that $(\mathrm{pr}_2)_*\mathcal{L} = M$ is invertible (locally free rank 1), as at each y we have constructed an open neighborhood V where $M|_V$ is trivial. Also, we have that $\mathrm{pr}_2^* M \cong \mathcal{L}$ as desired.

Finally, if $\mathcal{L} \cong \mathrm{pr}_2^* M'$ for some $M' \in \operatorname{Pic} Y$, then $M' \cong (\mathrm{pr}_2)_*\mathcal{L}$, so $M \cong M'$. □

Corollary 7.3. *Under the same hypotheses as the previous theorem, if for some $x \in X(k)$ we have that $\mathcal{L}|_{\{x\}\times Y} \cong \mathcal{O}_Y$, then $\mathcal{L}$ is trivial.*

Proof. We have that $\mathcal{L} = \mathrm{pr}_2^* M$ and $\mathrm{pr}_2 \circ (x \times \mathrm{id}_Y) = \mathrm{id}_Y$, so

$$\mathcal{O}_Y = \mathcal{L}|_{\{x\}\times Y} = (x \times \mathrm{id}_Y)^*\mathcal{L} \cong M, \tag{7.6}$$

so

$$\mathcal{L} \cong \mathrm{pr}_2^* M \cong \mathrm{pr}_2^* \mathcal{O}_Y \cong \mathcal{O}_{X\times Y}. \tag{7.7}$$

□

The above corollary tells us if $\mathcal{L}$ is trivial on all the "horizontal fibers", and is trivial on a *single* "vertical fiber", then $\mathcal{L}$ is trivial.

Sheet 3, question 11 tells us that we can have proper k-varieties X, Y and $\mathcal{L} \in \operatorname{Pic}(X \times_k Y)$ such that $L|_{x\times Y}$ and $\mathcal{L}|_{X\times y}$ are trivial for some $x \in X(k)$ and $y \in Y(k)$, but $\mathcal{L}$ is nontrivial. In particular, $\operatorname{Pic}(X \times Y) \ncong \operatorname{Pic}(X) \oplus \operatorname{Pic}(Y)$ in general. But we do have some nice stuff over 3 varieties, hence the theorem of the cube. This tells us that if a line bundle is trivial on the "faces" of a cube, it is trivial.

Theorem 7.4 (Theorem of the cube)**.** *Let X, Y, Z be varieties over k, X, Y proper. Let $x \in X(k)$, $y \in Y(k)$, $z \in Z(k)$. Let $\mathcal{L}$ be a line bundle on $X \times_k Y \times_k Z$. Assume that each of the line bundles $\mathcal{L}|_{x \times Y \times Z}, \mathcal{L}|_{X \times y \times Z}, \mathcal{L}|_{X \times Y \times z}$ is trivial. Then $\mathcal{L}$ is trivial.*

Remark 7.5.

1. In fact, Z can be a more general k-scheme. In particular, we don't need Z to be reduced.

2. The seesaw Theorem 7.2 also holds for more general Y.

Before we prove this, we use first prove a helper lemma.

Lemma 7.6. *Let V be a proper variety over an algebraically closed field k, let A be a finite local k-algebra, and let I be a one-dimensional ideal, so that $\dim_k I = 1$ and $I = k \cdot t \subset A$ for some t. Set $z = \operatorname{Spec} A$ and $z_1 = \operatorname{Spec} A/I$. Then we have an exact sequence which is functorial in V*

$$0 \to H^1(V, \mathcal{O}_V) \to \operatorname{Pic}(V \times z) \to \operatorname{Pic}(V \times z_1) \tag{7.8}$$

Proof. We have that $\operatorname{Pic}(X) = H^1(X, \mathcal{O}_X^\times)$ for any scheme X (take a trivializing cover of $\mathcal{L} \in \operatorname{Pic}(X)$, this gives a Čech cocycle).

Since I is one-dimensional, we must have that $I^2 = 0$ (as I^2 is an ideal properly contained in I). Thus for all $a, b \in I$, we have that $(1 + a)(1 + b) = 1 + (a + b)$, so we have an exact sequence of abelian groups

$$\begin{aligned} 0 \to I \to A^\times \to (A/I)^\times \to 0 \\ a \mapsto 1 + a \qquad\qquad \end{aligned} \tag{7.9}$$

Tensoring by $\mathcal{O}_V$, we get an exact sequence of sheaves of abelian groups

$$0 \to I\mathcal{O}_{V \times z} \to \mathcal{O}_{V \times z}^\times \to \mathcal{O}_{V \times z_1}^\times \to 0 \tag{7.10}$$

These sheaves live on $V \times z$ (we consider z_1 as a closed subscheme of z), which is homeomorphic to V. We also have an isomorphism of sheaves

$$\mathcal{O}_V \xrightarrow[\sim]{\cdot t} I\mathcal{O}_{V \times z} \tag{7.11}$$

given by multiplication by t. Applying the cohomology long exact sequence gives

$$0 \to H^0(V, \mathcal{O}_V) \to H^0(V \times z, \mathcal{O}_{V \times z}^\times) \to H^0(V \times z, \mathcal{O}_{V \times z_1}^\times) \tag{7.12}$$

Now, we have that $H^0(V, \mathcal{O}_V) = k$ because V is proper, and

$$H^0(V \times z, \mathcal{O}_{V \times z}^\times) = H^0(V \times z, \mathcal{O}_{V \times z})^\times = (H^0(V, \mathcal{O}_V) \otimes A)^\times = A^\times \tag{7.13}$$

by flat base change. Similarly, we have that $H^0(V \times z, \mathcal{O}_{V \times z_1}^\times) = (A/I)^\times$.

Thus the last map is surjective, so the map $H^0(V \times z, \mathcal{O}_{V \times z_1}^\times) \to H^1(V, \mathcal{O}_V)$ will be the zero map, and thus the following map will be injective. So we get an exact sequence

$$0 \to H^1(V, \mathcal{O}_V) \to \operatorname{Pic}(V \times z) \to \operatorname{Pic}(V \times z_1) \tag{7.14}$$

as desired. This map is clearly functorial in V by construction. □

Remark 7.7.

1. This lemma tells us that if we have a line bundle on $\operatorname{Pic}(V \times z_1)$ which lifts to a line bundle on $\operatorname{Pic}(V \times z)$, then the number of different lifts are given by $H^1(V, \mathcal{O}_V)$.

2. Take $A = k[\epsilon] = k[x]/(x^2)$, the ring of dual numbers, and $I = (\epsilon)$, so that $A/I = k$. Then $z = \operatorname{Spec} k[\epsilon]$ and $z_1 = \operatorname{Spec} k$. Then we have an exact sequence

$$0 \to H^1(V, \mathcal{O}_V) \to \operatorname{Pic}(V \times \operatorname{Spec} k[\epsilon]) \to \operatorname{Pic} V. \tag{7.15}$$

   The last map is the restriction to a closed subscheme, as $V \hookrightarrow V \times \operatorname{Spec} k[\epsilon]$ is a closed subscheme. But this has a retraction (a section? why?), so the map is surjective and we have a full exact sequence.

3. Recall that the Zariski tangent space at x is the set of k-morphisms $\operatorname{Spec} k(x)[\epsilon] \to X$ which restrict to the canonical map $\operatorname{Spec} k(x) \to X$ on the closed subscheme $\operatorname{Spec} k(x) \subset \operatorname{Spec} k(x)[\epsilon]$.

   This result then says that the tangent space to Pic is $H^1(V, \mathcal{O}_V)$. In fact, there is a group scheme $\operatorname{Pic}_{V/k}$ with $\operatorname{Pic}_{V/k}(k) = \operatorname{Pic}(V)$ and $\operatorname{Pic}_{V/k}(k[\epsilon) = \operatorname{Pic}(V \times \operatorname{Spec} k[\epsilon])$. So $H^1(V\mathcal{O}_V) = T_{\operatorname{Pic}_{V/k},0}$ and the lemma says that the tangent space is additive.

Now we are ready to prove the theorem.

Proof of Theorem 7.4. For simplicity, we will assume that $k = \overline{k}$. To prove this, we will consider other k-schemes Z. We will prove the theorem (a) where $Z = \operatorname{Spec} A$ and A is a finite local k-algebra, and then consider the case (b) where $Z = \operatorname{Spec} A$ and A is a Noetherian local k-algebra, and then consider the general case (c).

(a) Let $Z = \operatorname{Spec} A$, A a finite local k-algebra, so $z \in Z$ is a possibly non-reduced point. We proceed by induction on $\dim_k A \geq 1$. If $\dim A = 1$, then $Z = \operatorname{Spec} k$, so $X \times Y \times z \cong X \times Y \times Z$, so we are done.

Otherwise, there exists a one dimensional ideal $I \subset A$ with $\dim_k I = 1$, so $i = k \cdot t \subset A$. (Recall that $k = \overline{k}$, so we can't have $A =$ a finite field extension). Let $Z_1 = \operatorname{Spec} A/I$, and we have that $\dim_k A/I = \dim_k A - 1$. Thus we can assume by induction that $\mathcal{L}|_{X \times Y \times Z_1}$ is trivial. By the lemma, we have an exact sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^1(X \times Y, \mathcal{O}_{X \times Y} & \longrightarrow & \operatorname{Pic}(X \times Y \times Z) & \xrightarrow{\ c\ } & \operatorname{Pic}(X \times Y \times Z_1) \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow \\
0 & \longrightarrow & H^1(X, \mathcal{O}_X) \oplus H^1(Y, \mathcal{O}_Y) & \longrightarrow & \operatorname{Pic}(X \times Z) \oplus \operatorname{Pic}(Y \times Z) & \longrightarrow & \operatorname{Pic}(X \times Z_1) \oplus \operatorname{Pic}(Y \times Z_1)
\end{array}$$

The Künneth formula 5.10, and the fac that $H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y) = k$ because X and Y are proper, gives an isomorphism $H^1(\mathcal{O}_X) \oplus H^1(\mathcal{O}_Y) \cong H^1(X \times Y, \mathcal{O}_{X \times Y})$, which is an inverse to a. So a is an isomorphism. Recall that $\mathcal{L} \in \operatorname{Pic}(X \times Y \times Z)$, and by assumption we have that $b(\mathcal{L}) = c(\mathcal{L}) = 0$. Thus by a routine diagram chase we have that $\mathcal{L}$ is trivial.

(b) Now let $Z = \operatorname{Spec} A$, with $(A, \mathfrak{m}_A)$ a local Noetherian k-algebra. Let $Z_n = \operatorname{Spec} A/\mathfrak{m}_A^n$ for $n \geq 1$, then $A/\mathfrak{m}_A^n$ is finite, so by (a), $\mathcal{L}|_{X \times Y \times Z_n}$ is trivial for all n.

From base change, there exists finite modules Q, Q' such that for all $A \to B$,

$$H^0((X \times Y \times Z)_B, \mathcal{L}_B) = \mathrm{Hom}_A(Q, B)$$
$$H^0((X \times Y \times Z)_B, \mathcal{L}_B^\vee) = \mathrm{Hom}_A(Q', B) \tag{7.16}$$

As $\mathcal{L}|_{X \times Y \times z} \cong 0$ is trivial, we know that Q, Q' are acyclic (why?). Then for all $n \geq 1$, as $\mathcal{L}|_{X \times Y \times Z_n}$ is trivial, we have that $Q \otimes A/\mathfrak{m}_A^n \cong A/\mathfrak{m}_A^n$. So

$$\mathrm{ann}_A(Q) \subset \bigcap_n \mathfrak{m}_A^n = \{0\}, \tag{7.17}$$

so $Q \cong A \cong Q'$. So by the seesaw theorem 7.3, $\mathcal{L} \cong \mathcal{O}_{X \times Y \times Z}$ as $X \times Y$ is proper.

(c) Now, let Z be a variety. Then (b) implies that $\mathcal{L}|_{X \times Y \times \mathrm{Spec}\, \mathcal{O}_{Z,z}}$ is trivial. As $\mathrm{Spec}\, \mathcal{O}_{Z,z}$ contains the generic point of Z, and the set of $z' \in Z$ such that $\mathcal{L}|_{X \times Y \times z'}$ is trivial is closed, $\mathcal{L}|_{X \times Y \times z'}$ is trivial for all $z' \in Z$, so as $\mathcal{L}|_{x \times y \times Z}$ is trivial, $\mathcal{L}$ is trivial by the seesaw theorem 7.3. □

We give some nice corollaries. The following is the form of the Theorem of the Cube which we use most often, however it is not any easier to prove than the full thing.

Corollary 7.8. *Let X be an abelian variety over k, Y any k-scheme, and $f, g, h : Y \to X$ morphisms. Let $\mathcal{L}$ be a line bundle on X. Then*

$$\mathcal{M} = \mathcal{M}_{f,g,h} := (f+g+h)^*\mathcal{L} \otimes (f+g)^*\mathcal{L}^\vee \otimes (f+h)^*\mathcal{L}^\vee \otimes (g+h)^*\mathcal{L}^\vee \otimes f^*\mathcal{L} \otimes g^*\mathcal{L} \otimes h^*\mathcal{L} \tag{7.18}$$

is trivial, so congruent to $\mathcal{O}_Y$.

Proof. First consider $Y = X \times_k X \times_k X$, and $(f, g, h) = (\mathrm{pr}_1^3, \mathrm{pr}_2^3, \mathrm{pr}_3^3) : X \times X \times X \to X$. This is a sort of "universal case" of the problem. Then $\mathcal{M}|_{X \times X \times e} = q^* M$, where

$$q : X \times X \to X \times X \times X$$
$$(x, y) \mapsto (x, y, e) \tag{7.19}$$

We have that $(\mathrm{pr}_1^3 + \mathrm{pr}_2^3 + \mathrm{pr}_3^3) \circ q = (\mathrm{pr}_1^3 + \mathrm{pr}_2^3) \circ q = m$ as we send $(x, y) \mapsto x + y$. We can perform similar calculations for the other terms to determine that

$$\mathcal{M}|_{X \times X \times e} = e^*\mathcal{L} \cong \mathcal{O}_{X \times X \times e} \cong \mathcal{O}_{X \times X}. \tag{7.20}$$

So by symmetry, $\mathcal{M}|_{X \times e \times X} \cong \mathcal{M}|_{X \times X \times e} \cong \mathcal{O}_{X \times X}$, so by the theorem of the cube, $\mathcal{M} \cong \mathcal{O}_{X \times X \times X}$. For general $f, g, h : Y \to X$, we have that

$$\mathcal{M}_{f,g,h} = (f, g, h)^* \mathcal{M}|_{\mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3} \tag{7.21}$$

so this is trivial. □

We give two more consequences. The next theorem is better names the "theorem of the parallelogram", as we can draw a parallelogram with vertices $e, x, y, x + y$.

Theorem 7.9 (Theorem of the square)**.** *Let X be an abelian variety, $x, y \in X(k)$, $\mathcal{L} \in \mathrm{Pic}(X)$. Then*

$$T_{x+y}^*\mathcal{L} \cong T_x^*\mathcal{L} \otimes T_y^*\mathcal{L} \otimes \mathcal{L}^\vee. \tag{7.22}$$

Proof. Take $Y = X$ in Corollary 7.8, and take $f = x : X \to \operatorname{Spec} k \xrightarrow{x} X$, $g = y : X \to \operatorname{Spec} k \xrightarrow{y} X$, and $h = \mathrm{id}_X : X \to X$. Then $f + h = T_x$, $g + h = T_y$, $f + g + h = T_{x+y}$, $f + g$ is the constant function $x + y$, and if $p : X \to X$ is constant (so it factors through $\operatorname{Spec} k$, then $p^*\mathcal{L} \cong \mathcal{O}_X$). Then applying Corollary 7.8 gives the theorem. □

Corollary 7.10. *Let X be an abelian variety, and $n \in \mathbb{Z}$, and let $[n] : X \to X$ be the "multiplication by n" morphism. Let $\mathcal{L} \in \operatorname{Pic} X$. Then*

$$[n]^*\mathcal{L} \cong \mathcal{L}^{\otimes \frac{n(n+1)}{2}} \otimes (i^*\mathcal{L})^{\otimes \frac{n(n-1)}{2}} \tag{7.23}$$

where $i = [-1] : X \to X$ is the inverse morphism. In particular, if $\mathcal{L} \cong i^*\mathcal{L}$, so $\mathcal{L}$ is symmetric*, then*

$$[n]^*\mathcal{L} \cong \mathcal{L}^{\otimes n^2}. \tag{7.24}$$

If $\mathcal{L}^\vee \cong i^*\mathcal{L}$, so $\mathcal{L}$ is antisymmetric*, then*

$$[n]^*\mathcal{L} \cong \mathcal{L}^{\otimes n}. \tag{7.25}$$

Proof. We proceed by induction. Its a mess. Check both 0 and 1, as we need $n - 1$ and $n - 2$ to prove n. The same argument works for negative n. □

8 The Picard group of an abelian variety

$\mathrm{Pic}(X)$ is an abelian group under $\otimes_{\mathcal{O}_X}$ with inverse $\mathcal{L} \mapsto \mathcal{L}^\vee$, and unit $\mathcal{O}_X$. It is contravariant, so if $f : X \to Y$, then $f^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is a group homomorphism for all schemes X, Y. It is convenient to work over $\overline{k}$ sometimes. The next proposition tells us that we lose no information by doing so.

Proposition 8.1. *Let X be a proper k-variety. Then $\mathrm{Pic}(X) \hookrightarrow \mathrm{Pic}(X_{\overline{k}})$ is injective, where $X_{\overline{k}} = X \times_k \operatorname{Spec} \overline{k}$.*

Proof. Suppose we have $\mathcal{L} \in \mathrm{Pic}(X)$ such that $\mathcal{L}_{\overline{k}} \cong \mathcal{O}_{X_{\overline{k}}} \in \mathrm{Pic}(X_{\overline{k}})$ is trivial. We want to show that $\mathcal{L}$ is trivial, so $\mathcal{L} \cong \mathcal{O}_X$.

But $\mathcal{L} \cong \mathcal{O}_X$ if and only if $H^0(X, \mathcal{L}) \neq 0$ and $H^0(X, \mathcal{L}^\vee) \neq 0$ by Sheet 3, question 10 and this is true if and only if $H^0(X_{\overline{k}}, \mathcal{L}_{\overline{k}}) \neq 0$ and $H^0(X_{\overline{k}}, \mathcal{L}_{\overline{k}}^\vee) \neq 0$ by flat base change, and this is true if and only if $\mathcal{L}_{\overline{k}} \cong \mathcal{O}_{X_{\overline{k}}}$. □

Next we use the theorem of the square and other things to show that there exists a surjection $X(\overline{k}) \twoheadrightarrow \mathrm{Pic}^0(X_{\overline{k}})$ with finite kernel, where $\mathrm{Pic}^0(X_{\overline{k}})$ is a subgroup of $\mathrm{Pic}(X_{\overline{k}})$.

From now on, except when otherwise stated, assume that $k = \overline{k}$.

Proposition 8.2. *Let X be an abelian variety, and $\mathcal{L}$ a line bundle on X. We have that*

(i) For $x \in X(k)$, set $\varphi_{\mathcal{L}}(x) = T_x^*\mathcal{L} \otimes \mathcal{L}^\vee \in \mathrm{Pic}(X)$. Then $\varphi_{\mathcal{L}} : X(k) \to \mathrm{Pic}(X)$ is a homomorphism.

(ii) Set $K(\mathcal{L}) \coloneqq \ker \varphi_{\mathcal{L}} \subset X(k)$*. If we set*

$$\mathrm{Pic}^0(X) \coloneqq \{\mathcal{L} \in \mathrm{Pic}(X) \mid \varphi_{\mathcal{L}} = 0\} = \{\mathcal{L} \in \mathrm{Pic}(X) \mid \forall x \in X(k), T_x^*\mathcal{L} \cong \mathcal{L}\} \tag{8.1}$$

then $\mathrm{Pic}^0(X)$ *is a subgroup of* $\mathrm{Pic}(X)$*, and we set*

$$\mathrm{NS}(X) \coloneqq \mathrm{Pic}(X)/\mathrm{Pic}^0(X), \tag{8.2}$$

the Neron-Severi group *on* X*.*

Note that as $k = \overline{k}$, $X(k)$ is in bijection with the closed points of X. In general, define $\varphi_{\mathcal{L}} : X(\overline{k}) \to \mathrm{Pic}\, X_{\overline{k}}$ in the same way.

Proof. (i) We calculate $\varphi_{\mathcal{L}}(x+y) = \varphi_{\mathcal{L}}(x) + \varphi_{\mathcal{L}}(y)$ directly using the Theorem of the Square.

(ii) We have that

$$\begin{aligned}\varphi_{\mathcal{L}\otimes\mathcal{M}}(x) &= T_x^*(\mathcal{L}\otimes\mathcal{M}) \otimes (\mathcal{L}\otimes\mathcal{M})^\vee \\ &= \varphi_{\mathcal{L}}(x) + \varphi_{\mathcal{M}}(x)\end{aligned} \tag{8.3}$$

as $T_x^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ is a homomorphism.

So $\mathcal{L} \mapsto \varphi_{\mathcal{L}}$ is a homomorphism

$$\mathrm{Pic}(X) \to \mathrm{Hom}(X(k), \mathrm{Pic}(X)) \tag{8.4}$$

whose kernel is $\mathrm{Pic}^0(X)$. □

Definition 8.3. Let $\mathcal{L} \in \mathrm{Pic}(X)$. Define $\Lambda(\mathcal{L}) = m^*\mathcal{L} \otimes \mathrm{pr}_1^*\mathcal{L}^\vee \otimes \mathrm{pr}_2^*\mathcal{L}^\vee$, the *Mumford line bundle* on $X \times_k X$, where $m : X \times X \to X$ is the group law.

Proposition 8.4. *(i) For all* $x \in X(k)$*, we have that* $\Lambda(\mathcal{L})|_{X\times x} = \Lambda(\mathcal{L})|_{x\times X} = T_x^*\mathcal{L} \otimes \mathcal{L}^\vee = \varphi_{\mathcal{L}}(x)$ *and*

$$K(\mathcal{L}) = \{x \in X(k) \mid \Lambda(\mathcal{L})|_{X\times x} \cong \mathcal{O}_X.\} \tag{8.5}$$

(ii) $\mathcal{L} \in \mathrm{Pic}^0(X)$ *if and only if* $\Lambda(\mathcal{L}) \cong \mathcal{O}_{X\times_k X}$*.*

Proof. (i) Let $x \in X(k)$. Consider the map $(\mathrm{id}_X, x) : X \hookrightarrow X \times X$ with image $X \times x$. Then

$$\begin{aligned} m \circ (\mathrm{id}_X, x) &= T_x \\ \mathrm{pr}_1 \circ (\mathrm{id}_X, x) &= \mathrm{id}_X \\ \mathrm{pr}_2 \circ (\mathrm{id}_X, x) &= x \end{aligned} \tag{8.6}$$

so $\Lambda(\mathcal{L})|_{X\times x} = (\mathrm{id}_X, x)^*\Lambda(\mathcal{L}) = T_x^*\mathcal{L} \otimes \mathcal{L}^\vee \otimes \mathcal{O}_X^\vee = \Lambda(\mathcal{L})|_{x\times X}$ by symmetry, as X is commutative.

The second part of the claim follows by definitions.

(ii) We have that $\Lambda(\mathcal{L})_{e\times X} = \mathcal{O}_X$, so by the Seesaw Theorem 7.2, $\Lambda(\mathcal{L}) \cong \mathcal{O}_{X\times X}$ if and only if $\forall x \in X(k)$, $\Lambda(\mathcal{L})|_{X\times x} \cong \mathcal{O}_X$. This is true if and only if for all x, $\varphi_{\mathcal{L}}(x) = 0$, and this is true if and only if $\mathcal{L} \in \mathrm{Pic}^0(X)$ by definition. □

Proposition 8.5. *(i) For all* $\mathcal{L} \in \mathrm{Pic}(X)$*,* $\mathrm{im}\, \varphi_{\mathcal{L}} \subset \mathrm{Pic}^0(X)$*.*

(ii) If $\mathcal{L} \in \mathrm{Pic}^0(X)$, then $i^*\mathcal{L} \cong \mathcal{L}^\vee$, where $i^* = [-1] : X \to X$ is the inversion map.

Proof. (i)Let $\mathcal{M} = \varphi_{\mathcal{L}}(x) = T_x^*\mathcal{L} \otimes \mathcal{L}^\vee$, for $\mathcal{L} \in \mathrm{Pic}(X)$ and $x \in X(k)$. Then for all $y \in X(k)$,

$$\varphi_M(y) \cong \mathcal{O}_X \tag{8.7}$$

by the Theorem of the square (just calculate), so $M \in \mathrm{Pic}^0(X)$.

(ii) Let $\mathcal{L} \in \mathrm{Pic}^0(X)$. Then by Proposition 8.4,

$$m^*\mathcal{L} \cong \mathrm{pr}_1^* \mathcal{L} \otimes \mathrm{pr}_2^* \mathcal{L}. \tag{8.8}$$

Consider the "co-difference" morphism $d : X \to X \times X$ sending $x \mapsto (x, -x)$. We have that $m \circ d$ is the constant morphism $e : X \to X$, so

$$d^*m^*\mathcal{L} \cong \mathcal{O}_X. \tag{8.9}$$

Thus by (8.8), we have that

$$\mathcal{O}_X \cong d^*(\mathrm{pr}_1^* \mathcal{L} \otimes \mathrm{pr}_2^* \mathcal{L}) \cong \mathcal{L} \otimes i^*\mathcal{L}. \tag{8.10}$$

□

This looks like the elliptic curve group law. A good exercise is to work everything out an elliptic curve, as in that case the line bundles correspond to closed points. Our goal is to show the following.

Theorem 8.6. *If $\mathcal{L}$ is ample, then $K(\mathcal{L})$ is finite and*

$$\varphi_{\mathcal{L}} : X(k)/K(\mathcal{L}) \cong \mathrm{Pic}^0(X). \tag{8.11}$$

Proposition 8.7. *Let $\mathcal{L} \in \mathrm{Pic}(X)$, and $K(\mathcal{L}) \subset X(k)$. Then there exists a unique reduced closed subgroup scheme $Z \subset X$ such that $K(\mathcal{L}) = Z(k)$ and*

$$\Lambda(\mathcal{L})|_{X \times Z} = \mathcal{O}_{X \times Z} \tag{8.12}$$

Proof. Let

$$Z = \{x \in X \mid \Lambda(\mathcal{L})|_{X \times x} \cong \mathcal{O}_{X \times \mathrm{Spec}\, k(x)} \tag{8.13}$$

for all points $x \in X$ (not necessarily closed). Then $Z \subset X$ is a closed subset by Sheet 3, Question 10, and by Proposition 8.4, $K(\mathcal{L}) = Z(k)$. Give Z the reduced subscheme structure. By the Seesaw Theorem 7.2 on $X \times Z$, since $\Lambda(\mathcal{L})|_{e \times X} \cong \mathcal{O}_X \cong \Lambda(\mathcal{L})_{X \times x}$ for all $x \in Z(k)$, we have that $\Lambda(\mathcal{L})|_{X \times Z}$ is trivial. It remains to check that Z is a subgroup scheme.

As $k = \overline{k}$ and Z is reduced, $Z = \bigcup Z_i$ is a union of k-varieties, so $Z \times Z = \bigcup Z_i \times Z_j$ is reduced, as $Z_i \times Z_j$ is a variety (this does not work if k is not algebraically closed).

Consider the inclusion followed by the shear map:

$$Z \times Z \hookrightarrow X \times X \xrightarrow[\sim]{(m, \mathrm{pr}_2)} X \times X. \tag{8.14}$$

As (m, pr_2) is an isomorphism, and $Z(k) = K(\mathcal{L})$ is a subgroup of $X(k)$, the image of $Z \times Z$ under the above composite morphism is a reduced closed subscheme of $X \times X$ whose k-points are $Z(k) \times Z(k)$. So $Z \times Z \cong Z \times Z$ under the morphism $(x, y) \mapsto (m(x, y), x)$, so Z is a subgroup scheme as for all k-algebras R, $Z(R) \subset X(R)$ is a subgroup. □

Remark 8.8. (i) In fact, there exists a closed subgroup scheme $\underline{K}(\mathcal{L}) \subset X$ (not necessarily reduced), such that for all closed subschemes $S \subset X$,

$$\Lambda(\mathcal{L})|_{X\times S} \cong \mathcal{O}_{X\times S} \tag{8.15}$$

if and only if $S \subset \underline{K}(\mathcal{L})$. Further, we have that $Z = \underline{K}(\mathcal{L})^{\mathrm{red}}$. This is a sort of infinitesimal expansion of Z.

(ii) In particular, if $K(\mathcal{L})$ is infinite, then there exists a nonzero abelian subvariety $Y \subset X$ such that $Y(k) \subset X(k)$. We can take Y to be the irreducible component of Z containing e, and this will be an integral closed subgroup variety of proper X, hence an abelian variety.

Now, let $D \geq 0$ be an effective divisor on an abelian variety X/k with $k = \overline{k}$. So

$$D = \sum_{i=1}^{r} n_i D_i, \quad n_i \geq 0, \tag{8.16}$$

and each D_i is a codimension 1 subvariety (an integral closed subscheme). Because X is smooth over k, Weil divisors are the same thing as Cartier divisors. Define

$$H(D) = \{x \in X(k) \mid T_x^* D = D\} \tag{8.17}$$

where by $T_x^* D = D$ we mean that the divisors on both sides are the same. Note that $T_x^* D = T_{-x}(D)$. So as $\mathcal{O}_X(T_x^* D) = T_x^* \mathcal{O}_X(D)$, $H(D)$ is a subgroup of

$$K(\mathcal{O}_X(D)) = \{x \in X(k) \mid T_x^* \mathcal{O}_X(D) = \mathcal{O}_X(D)\}. \tag{8.18}$$

It turns out that like $K(\mathcal{L})$ (c.f Proposition 8.7), $H(D)$ is the k-points of a closed subscheme of X, for a much simpler reason. In fact, if $Y \subset X$ is any closed subset, and $x \in X(k)$, then $T_x(Y) = Y$ if and only if for all $y \in Y(k)$, $x + y \in Y(k)$ as the k-points are dense. This is true if and only if

$$\begin{aligned} x &\in \bigcap_{y\in Y(k)} \{z \in X \mid (z, y) \in m^{-1}(Y)\} \\ &= \bigcap_{y\in Y(k)} \mathrm{pr}_1(X \times \{y\} \cap m^{-1}(Y)), \end{aligned} \tag{8.19}$$

and this last set is closed because $X \times \{y\}$ and $m^{-1}(Y)$ are closed, so after projection we take the intersection of a bunch of closed sets.

Theorem 8.9. *Let D be an effective divisor on X, $\mathcal{L} = \mathcal{O}_X(D)$. The following are equivalent:*

(i) $\mathcal{L}$ is ample.

(ii) $K(\mathcal{L})$ is finite.

(iii) $H(D)$ is finite.

Proof. (ii) $\implies$ (iii) as $H(D) \subset K(\mathcal{O}_X(D))$.

(i) $\implies$ (ii): Suppose $\mathcal{L}$ is ample and $K(\mathcal{L})$ is infinite. Then $K(\mathcal{L}) \supset Y(k)$ for a nonzero abelian subvariety Y of X, as $K(\mathcal{L})$ is the closed points of a subgroup scheme (see Remark 8.8 (ii)). As

$\mathcal{L}|_Y$ is ample, we may assume $Y = X$. Then $K(\mathcal{L}) = X(k)$. So $\mathcal{L} \in \mathrm{Pic}^0(X) = \{\mathcal{L} \mid \varphi_{\mathcal{L}} = 0\}$ and so by Proposition 8.5 (ii), $i^*\mathcal{L} = \mathcal{L}^\vee$. So $\mathcal{L}^\vee$ is ample, so $\mathcal{L} \otimes \mathcal{L}^\vee = \mathcal{O}_X$ is ample. This means that X is a point, as $H^0(X, \mathcal{O}_X^{\otimes n}) = k$ for all n, and since $\mathcal{O}_X$ is ample, k generates $\mathcal{O}_X$, so $\dim X = 0$. This is a contradiction as we assume that $K(\mathcal{L})$ was infinite.

(iii) $\implies$ (i): Consider $\mathcal{L}^{\otimes 2} = \mathcal{O}_X(2D) \cong T_x^*\mathcal{L} \otimes T_{-x}^*\mathcal{L} = \mathcal{O}_X(T_x^*D + T_{-x}^*D)$ for all $x \in X(k)$ by the Theorem of the square 7.9. So for all $x \in X(k)$, there exists $s_x \in H^0(X, \mathcal{L}^{\otimes 2}) \setminus \{0\}$ such that $\mathrm{div}(s_x) = T_x^*D + T_{-x}^*D$ (the canonical section).

We want to construct a map $f : X \to \mathbb{P}^N$ given by $H^0(X, \mathcal{L}^{\otimes 2})$. For any $y \in X(k)$, $s_x(y) = 0$ if and only if $y \in T_x^*D \cup T_{-x}^*D$ if and only if one of $y \pm x \in D$ if and only if $x \in E_y = T_y^*D \cup i^*T_y^*D$. So for all $x \in (X \setminus E_y)(k)$, $s_x(y) \neq 0$. In particular, for all $y \in X(k)$, there exists $s \in H^0(X, \mathcal{L}^{\otimes 2})$ such that $s(y) \neq 0$. So $\mathcal{L}^{\otimes 2}$ gives a morphism $f : X \to \mathbb{P}_k^N$ such that $f^*\mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{L}^{\otimes 2}$.

We claim that f has finite closed fibers, so that $f^{-1}(p)$ is finite for all $p \in \mathbb{P}^N(k)$. Suppose $f(y) = f(y')$ for some $y, y' \in X(k)$. Then $\forall s \in H^0(X, \mathcal{L}^{\otimes 2})$, either $s(y) = 0 = s(y')$, or $s(y) \neq 0 \neq s(y')$. In particular, if $x \in E_y(k)$, then $s_x(y) = 0$ so $s_x(y') = 0$. Thus $E_y \subset E_{y'}$ (closed points are dense), so $E_y = E_y'$ as closed subsets of X by symmetry.

So if f does not have finite fibers, then there exists an irreducible $Y \subset f^{-1}(p)$, $p \in \mathbb{P}_k^N$ with $\dim Y > 0$ and such that for all $y, y' \in Y(k), E_y = E_{y'}$. This implies that for every irreducible component D' of D, $T_y^*D' = T_{y'}^*D'$ because $T_y^*D' = T_y^*D'$, and $E_y = E_{y'}$ so $T_y^*D \cup i^*T_y^*D = T_{y'}^*D \cup i^*T_{y'}^*D$ and Y is connected (**???**).

So $T_y^*D = T_{y'}^*D$, so $y - y' \in H(D)$, so $H(D)$ is infinite, a contradiction. Thus f has finite fibers. Our claim implies that $\mathcal{L}$ is ample by Zariski's main theorem, which says that if f is proper and has finite fibers, then f is finite.

Thus f is finite. It is a general fact that $f_*(\mathcal{F} \otimes f^*\xi) = f_*\mathcal{F} \otimes \xi$ for very general ξ. Since $f^*\mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{L}$, we have that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes 2n}) = H^p(\mathbb{P}^n, f_*\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(n)) = H^p(\mathbb{P}^n, (f_*\mathcal{F})(n))$ which vanishes if $p > 0$ and $n \gg 0$ for any coherent $\mathcal{F}$ on X, so $\mathcal{L}$ is ample by Serre's criterion 5.20. □

Corollary 8.10. *Any abelian variety over any base field k is projective.*

Proof. Let $U \subset X$ be any open affine, and let $D = X \setminus U$ with the reduced subscheme structure. By Sheet IV, D is a divisor. We want to show that $\mathcal{L} = \mathcal{O}_X(D)$ is ample. By Serre's criterion Proposition 5.20 (iv) and flat base change, it is enough to show that $\mathcal{L}_{\overline{k}}$ is ample on $X_{\overline{k}}$. So assume $k = \overline{k}$.

It is enough to show that $H(D)$ is finite. If $x \in H(D)$, then $T_x(D) = D$, so $T_x(U) = U$. So we may assume (translating if necessary) that $e \in U(k)$. This implies $H(D) \subset U(k)$ since $H(D)$ is a subgroup. If $H(D)$ is infinite, there exists a closed subscheme $Y \subset X$ of dimension greater than 0 with $Y(k) \subset H(D)$. So then $Y(k) \subset U(k)$, so $Y \subset U$ as the closed points are dense. As Y is proper of positive dimension and U is affine, this is impossible. □

Corollary 8.11. *Let X/k be an abelian variety and $n \geq 1$. Then* $\ker([n] : X(\overline{k}) \to X(\overline{k}))$ *is finite,* $[n] : X \to X$ *is surjective, and the group* $X(\overline{k})$ *is divisible.*

Proof. We can assume that $k = \overline{k}$. Suppose $\ker[n]$ is finite. If $x, x' \in X(k)$, then $[n](x) = [n](x')$ if and only if $x' - x \in \ker[n]$. So for all $y \in X(k)$, the fiber $[n]^{-1}(y)$ is finite. If we base change from k to $k(y)$, then the same holds for any $y \in X$, so $[n]$ has finite fibers, so as it is a morphism between varieties of the same dimension, it is dominant (the scheme-theoretic image of $[n]$ will be a subvariety of dimension equal to $\dim X$, hence dense). As X is proper, $[n]$ is proper, so $[n]$ is surjective (the image of X is a closed dense subset of X, which must be X).

Then for all $y \in X(k)$ and all $n \geq 1$, there exists $x \in X(k)$ with $nx = y$, so $X(k)$ is divisible.

Thus we just need to show that $\ker[n]$ is finite. Let $\mathcal{L} \in \mathrm{Pic}(X)$ be an ample line bundle. Replacing $\mathcal{L}$ by $\mathcal{L} \otimes i^*\mathcal{L}$ if necessary, we may assume that $\mathcal{L} \cong i^*\mathcal{L}$ is symmetric. If $\ker[n]$ is infinite, then there exists $V \subset X$ a subvariety of dimension $n > 0$ with $[n](V) = \{e\}$ and thus the composite map $V \hookrightarrow X \xrightarrow{[n]} X$ factors as $V \to \operatorname{Spec} k \xrightarrow{e} X$, so the restriction of $[n]^*\mathcal{L}$ to V is trivial. But $[n]^*\mathcal{L} \cong \mathcal{L}^{\otimes n^2}$ by Corollary 7.10, and $\mathcal{L}$ is ample, so $\mathcal{O}_V$ is ample, so $\mathcal{O}_V$ is affine and hence 0-dimensional as it is proper. □

This implies that $[n] : X \to X$ is a finite morphism (as in the proof of ???).

As $[n]$ is a morphism between smooth varieties of the same dimension, it is flat by miracle flatness.

So $[n]_*\mathcal{O}_X$ is therefore a finite flat $\mathcal{O}_X$-module, hence is locally free, whose rank is (by definition) the *degree* of n.

Theorem 8.12. *We have that* $\deg[n] = n^{2g}$*, where* $g = \dim X$*. In particular,* $\#\ker[n](\overline{k}) \leq n^{2g}$*.*

Proof. We use Hilbert polynomials. Assume that $\mathcal{L} \in \mathrm{Pic}(X)$ is symmetric and very ample, so it determines a closed immersion $X \hookrightarrow \mathbb{P}^N$. Consider $\mathcal{F} = [n]_*\mathcal{O}_X$, which is a coherent sheaf on X. What is its rank?

By Proposition 5.23, we have that

$$P(X, [n]_*\mathcal{O}_X, t) = \deg[n] P_X(t) + O\left(t^{g-1}\right) \tag{8.20}$$

and $\deg P_X = g = \dim X$. We have that $[n] : X \to X$ is finite, so for all $m \in \mathbb{Z}$,

$$H^0(X, [n]_*\mathcal{O}_X \otimes \mathcal{L}^m) = H^0(X, \mathcal{O}_X \otimes [n]^*\mathcal{L}^m) = H^0(X, \mathcal{L}^{\otimes mn^2}). \tag{8.21}$$

Thus for $m \gg 0$ we have that (**why??**)

$$\begin{aligned} \deg[n] P_X(m) &= P_x([n]_*\mathcal{O}_X, m) + O\left(m^{g-1}\right) \\ &= \dim H^0(X, [n]_*\mathcal{O}_X \otimes \mathcal{L}^m) + O\left(m^{g-1}\right) \qquad = \dim H^0(X, \mathcal{L}^{mn^2}) + O\left(m^{g-1}\right) \\ &= P_X(mn^2) + O\left(m^{g-1}\right). \end{aligned} \tag{8.22}$$

As $\deg P_X = g$, we have that $\deg[n] = n^{2g}$. □

In fact, it's easy now to show that if n is invertible in $k = \overline{k}$, then

$$\ker[n] \cong (\mathbb{Z}/n)^{2g} \tag{8.23}$$

the constant group scheme.

Theorem 8.13. *Let* $k = \overline{k}$ *and let* $\mathcal{L}$ *be an ample line bundle. Then*

$$\varphi_{\mathcal{L}} : X(k) \twoheadrightarrow \mathrm{Pic}^0(X) \tag{8.24}$$

is surjective.

The theorem tells us that $\mathrm{Pic}^0(X) \cong X(k)/K(\mathcal{L})$. Using this, one can give $\mathrm{Pic}^0(X)$ the structure of an abelian variety of the same dimension of X, the *dual abelian variety*.

Proof. Let $\mathcal{M} \in \mathrm{Pic}^0(X)$, and assume that $\mathcal{M} \notin \mathrm{im}\, \varphi_{\mathcal{L}}$. We will compute the cohomology of

$$\mathcal{F} = \Lambda(\mathcal{L}) \otimes \mathrm{pr}_1^* \mathcal{M}^\vee \in \mathrm{Pic}(X \times_k X) \tag{8.25}$$

in two different ways, by "slicing horizontally and vertically".

First we slice horizontally $x \in X(k)$. Then

$$\mathcal{F}|_{X \times x} = \Lambda(\mathcal{L})|_{X \times x} \otimes \mathcal{M}^\vee = T_x^* \otimes \mathcal{L}^\vee \otimes \mathcal{M}^\vee = \varphi_{\mathcal{L}}(x) \otimes \mathcal{M}^\vee \in \mathrm{Pic}^0(X). \tag{8.26}$$

By assumption (since $\mathcal{M} \notin \mathrm{im}\, \varphi_{\mathcal{L}}$), $\mathcal{F}|_{X \times x} \not\cong \mathcal{O}_X$. So by Sheet IV Question 7,

$$H^p(X, \mathcal{F}|_{X \times x}) = 0 \tag{8.27}$$

for all $p \geq 0$ and for all $x \in X(k)$.

This shows that for all open affines $U \subset X$, $H^p(X \times U, \mathcal{F}|_{X \times U}) = 0$ by Corollary 5.39 for the map $X \times U \to U$. So $H^p(X \times X, \mathcal{F}) = 0$ for all $p \geq 0$ by iterated Meyer-Vietoris (Sheet II Question 5).

Now we slice in the other vertically:

$$\mathcal{F}|_{x \times X} \cong \Lambda(\mathcal{L})|_{x \times X} = T_x^* \mathcal{L} \otimes \mathcal{L}^\vee \in \mathrm{Pic}^0(X). \tag{8.28}$$

If $x \in X(k) \setminus K(\mathcal{L})$, then $\mathcal{F}|_{x \times X} = \varphi_{\mathcal{L}}(x) \not\cong \mathcal{O}_X$ so by the same argument as in the horizontal case $H^p(X, \mathcal{F}|_{x \times X}) = 0$. For for all open affines $U \subset X \setminus K(\mathcal{L})$,

$$H^p(U \times X, \mathcal{F}|_{X \times X}) = 0 \tag{8.29}$$

for all $p \geq 0$.

As X is projective, there exists an open affine $V \subset X$ with $V \supset K(\mathcal{L})$. For instance, if $\dim X > 0$, let V be the complement of any hyperplane not meeting the finite set $K(\mathcal{L})$. If $\dim X = 0$ then everything is trivial.

By Sheet 2 Question 5 again, $H^p(V \times X, \mathcal{F}|_{V \times X}) = H^p(X \times X, \mathcal{F}) = 0$ for all $p \geq 0$. Then by Corollary 5.39 again, for all $x \in V(k)$, $H^p(X, \mathcal{F}|_{x \times X}) = 0$ for all $p \geq 0$. But $e \in K(\mathcal{L}) \subset V$, and $H^0(X, \mathcal{F}|_{e \times X}) = H^0(X, \mathcal{O}_X) = k$.

□