Discussion #34 4/27/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Problems

1. (a) Compute the Wronskian of the vector-valued functions

$$\mathbf{x}_1 = \begin{bmatrix} t \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} t^2 \\ 2t \end{bmatrix}.$$

Solution: We have

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = t \cdot 2t - t^2 = 2t^2 - t^2 = t^2.$$

(b) Are $\mathbf{x}_1$ and $\mathbf{x}_2$ linearly independent on $(0,1)$? On $\mathbf{R}$?

Solution: We have $W[\mathbf{x}_1, \mathbf{x}_2](t) = 0$ if and only if $t = 0$. Thus are vectors are linearly independent on every interval in $\mathbf{R}$ since we always have some $t_0 \in I$ such that

$$W[\mathbf{x}_1, \mathbf{x}_2](t_0) \neq 0.$$

(c) What conclusion can you draw about the coefficients in a system of homogeneous differential equations satisfied by $\mathbf{x}_1$ and $\mathbf{x}_2$?

Solution: The coefficients are not constant because the linearly independent solutions are not of the form

$$\mathbf{v}e^{\lambda t}$$

where $\mathbf{v} \in \mathbf{R}^2$.

(d) Find a system of homogeneous differential equations satisfied by $\mathbf{x}_1$ and $\mathbf{x}_2$ and verify your conclusion in part (c).

Solution: Consider

$$A(t) = \begin{bmatrix} 0 & 1 \\ 2/t^2 & -2/t \end{bmatrix}$$

then

$$A(t)\mathbf{x}_1(t) = \begin{bmatrix} 0 & 1 \\ 2/t^2 & -2/t \end{bmatrix} \cdot \begin{bmatrix} t \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{x}_1'(t)$$

and

$$A(t)\mathbf{x}_2(t) = \begin{bmatrix} 0 & 1 \\ 2/t^2 & -2/t \end{bmatrix} \cdot \begin{bmatrix} t^2 \\ 2t \end{bmatrix} = \begin{bmatrix} 2t \\ 2 \end{bmatrix} = \mathbf{x}_2'(t).$$

To find A we have a couple of options:

1. Grace.

2. Brute force.

Grace: We want

$$A(t)[\mathbf{x}_1 \quad \mathbf{x}_2] = [\mathbf{x}_1' \quad \mathbf{x}_2']$$

so

$$A(t) \cdot \begin{bmatrix} t & t^2 \\ 1 & 2t \end{bmatrix} = \begin{bmatrix} 1 & 2t \\ 0 & 2 \end{bmatrix}.$$

Now

$$\begin{aligned} A(t) &= \begin{bmatrix} 1 & 2t \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} t & t^2 \\ 1 & 2t \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 2t \\ 0 & 2 \end{bmatrix} \cdot \frac{1}{t^2} \begin{bmatrix} 2 & -t^2 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2t \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2/t^2 & -1 \\ -1/t^2 & 1/t^2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 2/t^2 & -2/t \end{bmatrix} \end{aligned}$$

Brute Force: Assume

$$A(t) = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}$$

then

$$\begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix} \cdot \begin{bmatrix} t \\ 1 \end{bmatrix} = \begin{bmatrix} a(t) \cdot t + b(t) \cdot 1 \\ c(t) \cdot t + d(t) \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

while

$$\begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix} \cdot \begin{bmatrix} t^2 \\ 2t \end{bmatrix} = \begin{bmatrix} a(t) \cdot t^2 + b(t) \cdot 2t \\ c(t) \cdot t^2 + d(t) \cdot 2t \end{bmatrix} = \begin{bmatrix} 2t \\ 2 \end{bmatrix}.$$

From the first rows of each vector equation we have

$$\begin{aligned} a(t) \cdot t + b(t) \cdot 1 &= 1 \\ a(t) \cdot t^2 + b(t) \cdot 2t &= 2t \end{aligned}$$

and so we can let

$$a(t) = 0 \quad \text{and} \quad b(t) = 1.$$

From the second rows of each vector equation we have

$$\begin{aligned} c(t) \cdot t + d(t) \cdot 1 &= 0 \\ c(t) \cdot t^2 + d(t) \cdot 2t &= 2 \end{aligned}$$

then for $t \neq 0$

$$c(t) = -\frac{d(t)}{t} \quad \text{and} \quad c(t) = \frac{2 - 2td(t)}{t^2}$$

and so

$$\begin{aligned} -td(t) &= 2 - 2td(t) \\ (-t + 2t)d(t) &= 2 \\ d(t) &= -\frac{2}{t} \end{aligned}$$

while

$$c(t) = -\frac{d(t)}{t} = \frac{2}{t^2}.$$

This gives

$$A(t) = \begin{bmatrix} 0 & 1 \\ 2/t^2 & -2/t \end{bmatrix}.$$

2. Part of a mixing device consists of two tanks connected by pipes. Suppose that initially, tank A contains 2 gallons of water with 1 pounds of salt dissolved in it, and tank B contains 2 gallons of pure water.

   Suppose that the mixer pipes fluid from A to B at the rate of 1 gallon per minute, and another pipe takes fluid from B to A at the same rate.

   (a) Set up a system of two first order equations and find the amount of salt in each tank as a function of time.

   Solution: Let $x(t)$ and $y(t)$ denote the amount of salt (in pounds) in tank A and tank B, respectively, at time t (in minutes), then it follows that

$$x(0) = 1, \quad y(0) = 0$$

   Each tank contains 2 gallons of liquid, and fluid flows at a rate of 1 gallon per minute in both directions where

$$\begin{aligned} x'(t) &= -\frac{1}{2}x(t) + \frac{1}{2}y(t) \\ y'(t) &= \frac{1}{2}x(t) - \frac{1}{2}y(t). \end{aligned}$$

   This gives the system

$$\mathbf{x}' = A\mathbf{x}$$

   where

$$A = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

We have

$$\begin{aligned}\det(A-\lambda I) &= \det\begin{bmatrix}-1/2-\lambda & 1/2\\ 1/2 & -1/2-\lambda\end{bmatrix}\\ &= \left(-\frac{1}{2}-\lambda\right)^2-\left(\frac{1}{2}\right)^2\\ &= \lambda^2+\lambda+\frac{1}{4}-\frac{1}{4}\\ &= \lambda^2+\lambda\\ &= \lambda(\lambda+1)\end{aligned}$$

and so our eigenvalues are $\lambda = 0$ and $\lambda = -1$.

Case $\lambda = 0$: We have

$$A-0I = \begin{bmatrix}-1/2 & 1/2\\ 1/2 & -1/2\end{bmatrix} \sim \begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}$$

and so

$$\mathbf{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}$$

is an eigenvector corresponding to eigenvalue -1.

With A symmetric we know that any eigenvector for $\lambda = -1$ is orthogonal to $(1,1)$, so we can pick

$$\mathbf{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}.$$

Our general solution is

$$\begin{aligned}x(t) &= c_1e^{0t}-c_2e^{-t} = c_1-c_2e^{-t}\\ y(t) &= c_1e^{0t}+c_2e^{-t} = c_1+c_2e^{-t}.\end{aligned}$$

From the initial conditions we can solve for c_1 and c_2

$$\begin{aligned}x(0) &= c_1-c_2 = 1\\ y(0) &= c_1+c_2 = 0\end{aligned}$$

and so

$$\left[\begin{array}{rr|r}1 & -1 & 1\\ 1 & 1 & 0\end{array}\right] \sim \left[\begin{array}{rr|r}1 & -1 & 1\\ 0 & 2 & -1\end{array}\right] \sim \left[\begin{array}{rr|r}1 & 0 & 1/2\\ 0 & 1 & -1/2\end{array}\right]$$

tells us

$$c_1 = \frac{1}{2} \quad \text{and} \quad c_2 = -\frac{1}{2}.$$

The particular solution is

$$\begin{aligned}x(t) &= \frac{1}{2}+\frac{1}{2}e^{-t}\\ y(t) &= \frac{1}{2}-\frac{1}{2}e^{-t}.\end{aligned}$$

(b) Will tank B ever have more salt than tank A? What happens to the two tanks as $t \to \infty$?

Solution: We want to figure out if

$$x(t) < y(t)$$

for some $t > 0$. Notice that

$$\frac{1}{2}e^{-t} > -\frac{1}{2}e^{-t}$$

so it follows that

$$x(t) > y(t)$$

for all $t \geq 0$. Thus tank A is always saltier than tank B. As $t \to \infty$,

$$x(t) \to \frac{1}{2} \quad \text{and} \quad y(t) \to \frac{1}{2}$$

so the two tanks both stabilize to having 1/2 pound of salt each.

3. Find the real-valued general solution of the system

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

for

$$A = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}.$$

Solution: We have eigenvalues

$$\begin{aligned} \det(A - I\lambda) &= \begin{vmatrix} -2-\lambda & -5 \\ 1 & 2-\lambda \end{vmatrix} \\ &= (-2-\lambda)(2-\lambda) - (-5)\cdot 1 \\ &= -4 + \lambda^2 + 5 \\ &= \lambda^2 + 1 \\ &= (\lambda + i)(\lambda - i) \end{aligned}$$

of $\pm i$.

Case $\lambda = i$: We

$$A - iI = \begin{bmatrix} -2-i & -5 \\ 1 & 2-i \end{bmatrix} \sim \begin{bmatrix} 1 & 2-i \\ 0 & 0 \end{bmatrix}$$

tells us an eigenvector corresponding to i is

$$\mathbf{v}_1 = \begin{bmatrix} -2+i \\ 1 \end{bmatrix}.$$

(Your eigenvector may look different if you choose its first component to be a **R**-valued.) An eigenvector for $\lambda = -i$ is therefore

$$\mathbf{v}_2 = \begin{bmatrix} -2-i \\ 1 \end{bmatrix}$$

so our general solution is a linear combination of

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} \cos(t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(t) = \begin{bmatrix} -2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix}$$

and

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} \sin(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(t) = \begin{bmatrix} \cos(t) - 2\sin(t) \\ \sin(t) \end{bmatrix}.$$

We can express this as

$$\begin{bmatrix} -2\cos(t) - \sin(t) & \cos(t) - 2\sin(t) \\ \cos(t) & \sin(t) \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

4. Solve the following differential equations:

(a) $y' = 3y$

Solution: We separate variables

$$\frac{1}{y} dy = 3\ dx,$$

integrate both sides

$$\ln|y| = 3x + C,$$

then solve for y

$$y = e^{3x+C}$$
$$y = C_0 e^{3x}$$

where $C_0 \in \mathbf{R}$.

(b) $y' = xy$

Solution: We separate variables

$$\frac{1}{y} dy = x\ dx,$$

integrate both sides

$$\ln|y| = \frac{1}{2}x^2 + C,$$

then solve for y

$$\begin{aligned} y &= e^{1/2x^2+C} \\ y &= C_0 e^{1/2x^2} \end{aligned}$$

where $C_0 \in \mathbf{R}$.

(c) $y' + (x - \sigma)y = 0$

Solution: We separate variables

$$\frac{1}{y} dy = (\sigma - x)\, dx,$$

integrate both sides

$$\ln|y| = \sigma x - \frac{1}{2}x^2 + C,$$

then solve for y

$$\begin{aligned} y &= e^{\sigma x - \frac{1}{2}x^2 + C} \\ y &= C_0 e^{\sigma x - x^2/2} \end{aligned}$$

where $C_0 \in \mathbf{R}$.

5. Evaluate the following partial derivatives:

(a) $\frac{\partial}{\partial x}\left(x^2 y + \cos(y)e^x - 2\sin(x)\right)$

Solution: We have

$$\frac{\partial}{\partial x}\big(x^2 y + \cos(y)e^x - 2\sin(x)\big) = 2xy + \cos(y)e^x - 2\cos(x).$$

(b) $\frac{\partial}{\partial t}\left(\cos(tx)\right)\Big|_{t=0}$

Solution: We have

$$\frac{\partial}{\partial t}\left(\cos(tx)\right) = -\sin(tx)x$$

and so

$$\frac{\partial}{\partial t}\left(\cos(tx)\right)\Big|_{t=0} = -\sin(0)x = 0.$$

(c) $\frac{\partial}{\partial y}\left(x+3\right)$

Solution: We have

$$\frac{\partial}{\partial y}\left(x+3\right) = 0$$

(d) $\dfrac{\partial}{\partial y}\,(y^3 - 3\cos(2y))$

Solution: We have

$$\frac{\partial}{\partial y}\left(y^3 - 3\cos(2y)\right) = 3y^2 + 6\sin(2y).$$