Discussion #33 4/24/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Problems

1. (a) Show

$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t},$$

where $c_1, c_2 \in \mathbf{R}$ satisfies

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

Solution: We have

$$\mathbf{x}' = 3c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} - c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t},$$

while

$$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} = - \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \cdot \mathbf{x} = 3c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} - c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$

since the functions e^{3t} and e^{-t} are scalar-valued.

(b) Find the eigenvalues of

$$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}.$$

Solution: As shown in (a), the eigenvalues are 3 and -1. One can also show

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) - 1 \cdot 4 = (\lambda+1)(\lambda-3).$$

(c) Explain where the solution in (a) came from.

Solution: We take the eigenvector for each eigenvalue λ, and then rescale it by the function $e^{\lambda t}$, and then add up all of the terms

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t}.$$

(d) Find c_1 and c_2 such that

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Solution: We have

$$\mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^0 + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^0 = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -4 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3/4 \\ 0 & 1 & 1/4 \end{array}\right]$$

tells us

$$\mathbf{x}(t) = \frac{3}{4}\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + \frac{1}{4}\begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.$$

2. Consider the second order differential equation $y'' + 3y' + 4y = 0$.

   (a) Let $x_1 = y$ and $x_2 = y'$. Rewrite the equation as a system of two first order equations in the two functions x_1 and x_2.

   Solution: We have

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -4x_1 - 3x_2 \end{aligned}$$

   (b) Show that the system in (a) can be written as $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix}.$$

   Solution: This follows immediately from (a).

   (c) Show that the equation $y''' + 2y'' - y' - 2y = 0$ can also be written as $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}.$$

   Solution: We let

$$x_1 = y, \quad x_2 = y', \quad \text{and} \quad x_3 = y''$$

   then

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= 2x_1 + x_2 - 2x_3 \end{aligned}$$

   so

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}.$$

3. Write

$$\left(1 - t^2\right) y'' - 2ty' + 2y = 0$$

in its normal form.

Solution: Let $x_1 = y$ and $x_2 = y'$ then

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{2}{1-t^2}x_1 - \frac{2t}{1-t^2}x_2 \end{aligned}$$

so

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{2}{1-t^2} & \frac{2t}{1-t^2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

4. Consider the vectors

$$\mathbf{x}_1(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}, \quad \mathbf{x}_2(t) = \begin{bmatrix} e^t \\ 2t \end{bmatrix}.$$

   (a) Compute the Wronskian of $\mathbf{x}_1$ and $\mathbf{x}_2$.

   Solution: We have

$$W[\mathbf{x}_1, \mathbf{x}_2](t) \begin{vmatrix} t & e^t \\ 1 & 2t \end{vmatrix} = t \cdot 2t - e^t \cdot 1 = 2t^2 - e^t$$

   (b) Are $\mathbf{x}_1$ and $\mathbf{x}_2$ linearly independent on $\mathbf{R}$?

   Solution: The vectors are linearly independent on $\mathbf{R}$ since

$$2 \cdot 1^2 - e < 0$$

   and thus $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$ for some $t \in \mathbf{R}$.

5. Define $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, and $\mathbf{x}_3(t)$, for $-\infty < t < \infty$, by

$$\mathbf{x}_1(t) = \begin{bmatrix} \sin(t) \\ \sin(t) \\ 0 \end{bmatrix}, \quad \mathbf{x}_2(t) = \begin{bmatrix} \sin(t) \\ 0 \\ \sin(t) \end{bmatrix}, \quad \mathbf{x}_3(t) = \begin{bmatrix} 0 \\ \sin(t) \\ \sin(t) \end{bmatrix}$$

   (a) Show that for the three scalar functions *in each individual row* there are nontrivial linear combinations that sum to zero for all t.

   Solution: In the first row consider:

$$\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_3.$$

   In the second row consider:

$$\mathbf{x}_1 + \mathbf{x}_2 - \mathbf{x}_3.$$

   In the third row consider:

$$\mathbf{x}_1 + \mathbf{x}_2 - \mathbf{x}_3.$$

(b) Show that, nonetheless, the three vector functions are linearly independent. (No single nontrivial combination works for *each row, for all t*.)

Solution: Assume $\sin(t) \neq 0$, then

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

tells us the column vectors are linearly independent on $\mathbf{R}$.

(c) Calculate the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3](t)$.

Solution: We have

$$\begin{aligned}
\begin{vmatrix} \sin(t) & \sin(t) & 0 \\ \sin(t) & 0 & \sin(t) \\ 0 & \sin(t) & \sin(t) \end{vmatrix} &= \sin(t) \begin{vmatrix} 0 & \sin(t) \\ \sin(t) & \sin(t) \end{vmatrix} - \sin(t) \begin{vmatrix} \sin(t) & \sin(t) \\ 0 & \sin(t) \end{vmatrix} \\
&\quad + 0 \cdot \begin{vmatrix} \sin(t) & 0 \\ 0 & \sin(t) \end{vmatrix} \\
&= \sin(t)(0 \cdot \sin(t) - \sin^2(t)) - \sin(t)(\sin^2(t) - 0) \\
&= -2\sin^3(t).
\end{aligned}$$