Discussion #31 4/20/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Problems

1. Answer the following *true* or *false*:
   (a) The functions $y_1(t) = e^{\alpha t}\cos(\beta t)$ and $y_2(t) = e^{\alpha t}\sin(\beta t)$ are linearly independent.

Solution: True:

Direct Approach: Choose $k \in \mathbf{R}$ such that

$$e^{\alpha t}\cos(\beta t) = k e^{\alpha t}\sin(\beta t)$$

for all t. Then

$$\cos(\beta t) = k\sin(\beta t)$$

for all t. If $t = 0$ we have

$$1 = 0$$

and this cannot hold. It follows that the functions are linearly independent.

Wronskian Approach (This only works to show linear independence on $\mathbf{R}$ if the Wronksian is nonzero at one point in $\mathbf{R}$.): We have

$$\begin{aligned} y_1'(t) &= \alpha e^{\alpha t}\cos(\beta t) - \beta e^{\alpha t}\sin(\beta t) = e^{\alpha t}(\alpha\cos(\beta t) - \beta\sin(\beta t)) \\ y_2'(t) &= \alpha e^{\alpha t}\sin(\beta t) + \beta e^{\alpha t}\cos(\beta t) = e^{\alpha t}(\alpha\sin(\beta t) + \beta\cos(\beta t)) \end{aligned}$$

and so

$$W[y_1, y_2](t) = \begin{vmatrix} e^{\alpha t}\cos(\beta t) & e^{\alpha t}\sin(\beta t) \\ e^{\alpha t}(\alpha\cos(\beta t) - \beta\sin(\beta t)) & e^{\alpha t}(\alpha\sin(\beta t) + \beta\cos(\beta t)) \end{vmatrix}$$

In particular

$$W[y_1, y_2](0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

immediately shows the functions are linearly independent because we showed the Wronskian is not identically 0. If instead, we compute the Wronskian for all t, we find

$$\begin{aligned} W[y_1, y_2](t) &= e^{2\alpha t}\begin{vmatrix} \cos(\beta t) & \sin(\beta t) \\ \alpha\cos(\beta t) - \beta\sin(\beta t) & \alpha\sin(\beta t) + \beta\cos(\beta t) \end{vmatrix} \\ &= e^{2\alpha t}\left(\cos(\beta t)\cdot(\alpha\sin(\beta t) + \beta\cos(\beta t)) - \sin(\beta t)\cdot(\alpha\cos(\beta t) - \beta\sin(\beta t))\right) \\ &= e^{2\alpha t}\left(\alpha\sin(\beta t)\cos(\beta t) + \beta\cos^2(\beta t) - \alpha\sin(\beta t)\cos(\beta t) + \beta\sin^2(\beta t)\right) \\ &= e^{2\alpha t}\left(\beta(\cos^2(\beta t) + \sin^2(\beta t))\right) \\ &= \beta e^{2\alpha t} \\ &\neq 0 \end{aligned}$$

for all t. Either way, the two functions are linearly independent on $\mathbf{R}$.

Remember, if we have a Wronskian that is identically zero for all t, and we do not know if the functions form a fundamental solution set for some nth order linear differential equation, we **cannot** assert linear independence. The situation is ambiguous.

(b) The polynomial $r^4 - 2$ has precisely two distinct roots.

Solution: False: The polynomial

$$r^4 - 2 = \left(r^2 - \sqrt{2}\right)\left(r^2 + \sqrt{2}\right)$$

has 4 distinct roots over $\mathbf{C}$, two of which are real.

(c) The functions

$$y_1 = t\sin(t) \quad \text{and} \quad y_2 = \cos(t)$$

are linearly independent.

Solution: True:

Direct Approach: Choose $k \in \mathbf{R}$ such that

$$\cos(t) = kt\sin(t)$$

for all t. Then for $t = 0$ we find

$$1 = 0$$

and so no k can exist. Thus the two functions are linearly independent.

Wronskian Approach (This only works to show linear independence on $\mathbf{R}$ if the Wronksian is nonzero at one point in $\mathbf{R}$.):

We have

$$\begin{aligned} y_1'(t) &= \frac{d}{dt}\left(t\sin(t)\right) = \sin(t) + t\cos(t) \\ y_2'(t) &= -\sin(t) \end{aligned}$$

gives us

$$W[y_1, y_2](t) = \begin{vmatrix} t\sin(t) & \cos(t) \\ \sin(t) + t\cos(t) & -\sin(t) \end{vmatrix}$$

and notice that

$$W[y_1, y_2](\pi/2) = \begin{vmatrix} \pi/2 & 0 \\ 1 & -1 \end{vmatrix} = -\pi/2 \neq 0$$

immediately shows the functions are linearly independent because we showed the Wronskian is not identically 0. If instead, we compute the Wronskian for all t, we find

$$\begin{aligned} W[y_1, y_2](t) &= \begin{vmatrix} t\sin(t) & \cos(t) \\ \sin(t) + t\cos(t) & -\sin(t) \end{vmatrix} \\ &= t\sin(t) \cdot (-\sin(t)) - \cos(t) \cdot (\sin(t) + t\cos(t)) \\ &= -t\sin^2(t) - \cos(t)\sin(t) - t\cos^2(t) \\ &= -t(\sin^2(t) + \cos^2(t)) - \cos(t)\sin(t) \\ &= -t \cdot 1 - \cos(t)\sin(t) \\ &= -t - \cos(t)\sin(t) \end{aligned}$$

and so again

$$W[y_1, y_2](\pi/2) \neq 0.$$

Either way, the two functions are linearly independent.

2. Find a particular solution of:
   (a) $y'' + 4y = 3x^3$

   Solution: Our homogeneous solution will not have a solution of the form

   $$y_h = 3x^3$$

   since our characteristic polynomial

   $$r^2 + 4$$

   has imaginary roots. Then

   $$\begin{aligned} y_p &= A_3x^3 + A_2x^2 + A_1x + A_0 \\ y_p' &= 3A_3x^2 + 2A_2x + A_1, \\ y_p'' &= 6A_3x + 2A_2. \end{aligned}$$

   and so

   $$\begin{aligned} y_p'' + 4y_p &= (6A_3x + 2A_2) + 4(A_3x^3 + A_2x^2 + A_1x + A_0) \\ &= 4A_3x^3 + 4A_2x^2 + (6A_3 + 4A_1)x + (2A_2 + 4A_0) \\ &= 3x^3. \end{aligned}$$

   Then it follows that

   $$\begin{aligned} 4A_0 + 2A_2 &= 0, & 4A_1 + 6A_3 &= 0, \\ 4A_2 &= 0, & 4A_3 &= 3 \end{aligned}$$

and

$$\left[\begin{array}{cccc|c} 4 & 0 & 2 & 0 & 0 \\ 0 & 4 & 0 & 6 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1/2 & 0 & 0 \\ 0 & 4 & 0 & 6 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 3/2 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -9/8 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3/4 \end{array}\right]$$

gives solution $A_3 = 3/4$, $A_2 = 0$, $A_1 = -9/8$, and $A_0 = 0$. Hence a particular solution is

$$y_p(x) = \frac{3}{4}x^3 - \frac{9}{8}x.$$

(b) $y'' + 4y = \cos(x)$

Solution: Our homogeneous solution will not have a solution of the form

$$y_h = \cos(x)$$

since our characteristic polynomial

$$r^2 + 4$$

has imaginary roots $r = \pm 2i$. Then

$$\begin{aligned} y_p &= A\cos(x) + B\sin(x) \\ y_p' &= -A\sin(x) + B\cos(x) \\ y_p &= -A\cos(x) - B\sin(x) \end{aligned}$$

gives

$$\begin{aligned} y_p'' + 4y_p &= -A\cos(x) - B\sin(x) + 4A\cos(x) + 4B\sin(x) \\ &= 3A\cos(x) + 3B\sin(x) \\ &= \cos(x) \end{aligned}$$

and so

$$A = 1/3 \quad \text{and} \quad B = 0.$$

Hence a particular solution is

$$y_p(x) = \frac{1}{3}\cos(x).$$

(c) $y'' + 4y = \cos(2x)$

Solution: The term

$$\cos(2x)$$

is a homogeneous solution and and we should not use it as our guess for the particular solution. All we have to do is rescale $\cos(2x)$ by x^s where $s = 1$,

$$\begin{aligned} y_p &= Ax\cos(2x) + Bx\sin(2x) \\ y_p' &= -2Ax\sin(2x) + A\cos(2x) + B\sin(2x) + 2Bx\cos(2x) \\ y_p'' &= -4A\sin(2x) - 4Ax\cos(2x) - 4Bx\sin(2x) + 4B\cos(2x) \\ &= -4(\sin(2x)(A + Bx) + \cos(2x)(Ax - B)) \end{aligned}$$

then

$$\begin{aligned} y_p'' + 4y_p &= -4(\sin(2x)(A + Bx) + \cos(2x)(Ax - B)) + 4(Ax\cos(2x) + Bx\sin(2x)) \\ &= -4A\sin(2x) + 4B\cos(2x) \\ &= \cos(2x) \end{aligned}$$

Thus

$$A = 0 \quad \text{and} \quad B = 1/4$$

so a particular solution is

$$y_p(x) = \frac{1}{4}x\sin(2x).$$

3. Find the general form of a particular solution, but do **not** determine the values of the coefficients, of:

(a) $y'' + 2y' + y = e^{-t} + \cos(t)$.

Solution: With a repeated root of $t = -1$ in our characteristic polynomial

$$r^2 + 2r + 1$$

we choose rescale e^{-t} by t^s where $s = 2$ in our guess for the particular solution. Thus

$$y_p = At^2e^{-t} + B\cos(t) + C\sin(t)$$

where $A, B, C \in \mathbf{R}$.

(b) $y^{(4)} + 5y'' + 4y = \sin(x) + \cos(2x)$

Solution: Our characteristic polynomial is

$$r^4 + 5r + 4 = (r^2 + 4)(r^2 + 1)$$

and so our homogeneous solution is of the form

$$y_h = c_1 \cos(x) + c_2 \sin(x) + c_3 \cos(2x) + c_4 \sin(2x).$$

Then since both $\sin(x)$ and $\cos(2x)$ are in the homogeneous solution, we choose

$$y_p = Ax\cos(x) + Bx\sin(x) + Cx\cos(2x) + Dx\sin(2x)$$

as our particular solution form.

(c) $y^{(4)} - 2y'' + y = x^2 \cos(x)$

Solution: Our characteristic polynomial is

$$r^4 - 2r^2 + 1 = (r-1)^2(r+1)^2$$

and so our homogeneous solution is of the form

$$y_h = c_1 e^x + c_2 x e^x + c_3 e^{-x} + c_4 x e^{-x}.$$

We have a 2nd order polynomial multiplied by cosine. This means we use

$$y_p = \left(A_0 + A_1 x + A_2 x^2\right)\cos(x) + \left(B_0 + B_1 x + B_2 x^2\right)\sin(x)$$

as our particular solution form.

4. Solve the initial value problem

$$y'' + 4y = 2x, \quad y(0) = 1, \quad \text{and} \quad y'(0) = 2.$$

Solution: Our characteristic polynomial is

$$r^2 + 4$$

and so our homogeneous solution is of the form

$$y_h = c_1 \cos(2x) + c_2 \sin(2x).$$

Then

$$y_p = A_0 + A_1 x$$

is our guess for the particular solution. Notice

$$y_p' = A_1 \quad \text{and} \quad y_p'' = 0$$

implies

$$y_p'' + 4y_p = 4A_0 + 4A_1 = 2x$$

and thus

$$A_0 = 0 \quad \text{and} \quad A_1 = 1/2.$$

Thus our general solution is

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{x}{2}.$$

and

$$y' = -2c_1 \sin(2x) + 2c_2 \cos(2x) + 1/2$$

gives

$$\begin{aligned} y(0) &= c_1 = 1 \\ y'(0) &= 2c_2 + 1/2 = 2. \end{aligned}$$

Thus

$$c_1 = 1 \quad \text{and} \quad c_2 = 3/4$$

and our solution is

$$y = \cos(2x) + \frac{3}{4}\sin(2x) + \frac{x}{2}.$$