Discussion #30 4/17/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Problems

1. Find the real-valued, general solution in t of the following:

   (a) $y'' + 2y' = 0$

   Solution: Our characteristic polynomial is

   $$c(r) = r^2 + 2r = r(r+2)$$

   and so its roots are $r = -2$ and $r = 0$. Thus

   $$y(t) = c_1 e^{0t} + c_2 e^{-2t} = c_1 + c_2 e^{-2t}$$

   is our general solution.

   (b) $y'' - y' - 12y = 0$

   Solution: Our characteristic polynomial is

   $$c(r) = r^2 - r - 12 = (r+3)(r-4)$$

   and so its roots are $r = -3$ and $r = 4$. Thus

   $$y(t) = c_1 e^{-3t} + c_2 e^{4t}$$

   is our general solution.

   (c) $y'' - 10y' + 25y = 0$

   Solution: Our characteristic polynomial is

   $$c(r) = r^2 - 10r + 25 = (r-5)(r-5)$$

   and so its repeated root is 5. Thus

   $$y(t) = c_1 e^{5t} + c_2 t e^{5t}$$

   is our general solution.

   (d) $y'' + 9y = 0$

   Solution: Our characteristic polynomial is

   $$c(r) = r^2 + 9 = (r+3i)(r-3i)$$

   and so its roots are $r = \pm 3i$. Thus

   $$y(t) = c_1 \cos(3t) + c_2 \sin(3t)$$

   is our general solution.

2. Find a fundamental set $\{y_1, y_2\}$ of solutions to the equation $y'' + 4y = 0$ such that

$$y_1(\pi/2) = 1, \quad y_2(\pi/2) = 0,$$
$$y_1'(\pi/2) = 0, \quad y_2'(\pi/2) = 1$$

Is your set unique?

Solution: Our characteristic polynomial is

$$c(r) = r^2 + 9 = (r + 2i)(r - 2i)$$

and so its roots are $r = \pm 2i$. Thus

$$y_1 = a\cos(2t) \quad \text{and} \quad y_2 = b\sin(2t)$$

form a fundamental solution set. Notice

$$y_1' = -2a\sin(2t) \quad \text{and} \quad y_2 = 2b\cos(2t)$$

so

$$y_1(\pi/2) = -a, \quad y_2(\pi/2) = 0,$$
$$y_1'(\pi/2) = 0, \qquad y_2'(\pi/2) = -2b$$

we find

$$a = -1 \quad \text{and} \quad -2b = 1$$

and thus

$$a = -1 \quad \text{and} \quad b = -1/2.$$

There is only one set of fundamental solutions with the desired properties:

$$\{-\cos(2t), -1/2\sin(2t)\}.$$

3. (a) If a, b, and c are positive constants, show that all solutions of $ay'' + by' + cy = 0$ approach zero as $t \to \infty$.

   Solution: If we have positive constants, then the polynomial

$$P(r) = ar^2 + br + c$$

   must have real roots, repeated roots, or complex roots given by

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

With $a, b, c > 0$, then

$$b^2 > b^2 - 4ac$$

and so

$$b = |b| > \sqrt{b^2 - 4ac}.$$

This tells us that the real part of our roots must be negative because

$$-b \pm \sqrt{b^2 - 4ac} < 0 \quad \text{and} \quad 2a > 0.$$

Thus we have a solution of the form

$$y = c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$$

where $r_1, r_2 > 0$, or

$$y = c_1 e^{-\lambda t} + c_2 t e^{-\lambda t}$$

where $\lambda > 0$, or finally

$$y = c_1 e^{-\alpha t} \cos(\beta t) + c_2 e^{-\alpha t} \sin(\beta t)$$

where $\alpha > 0$.

In each case

$$y \to 0$$

as $t \to \infty$.

(b) If $a > 0$, $c > 0$, and $b = 0$, show that the result of part (a) is not true, but that all solutions are bounded as $t \to \infty$.

Solution: Here the characteristic polynomial is of the form

$$P(r) = ar^2 + c = 0,$$

and with $a, c > 0$ we have purely imaginary roots of the form

$$r = \pm i\sqrt{c/a}.$$

Thus the general solution is of the form

$$y = c_1 \cos\left(\sqrt{\frac{c}{a}} t\right) + c_2 \sin\left(\sqrt{\frac{c}{a}} t\right).$$

These functions are bounded, but not convergent as $t \to \infty$.

(c) Now suppose that $a > 0$, $b > 0$, but that $c = 0$. Show that the result of part (a) is not true, but that all solutions approach a constant that depends on the initial conditions as $t \to \infty$. Determine this constant for the initial conditions $y(0) = y_0$, $y'(0) = y'_0$.

Solution: We have characteristic polynomial

$$ar^2 + br = r(ar + b)$$

with roots $r = 0$ and $r = -b/a$. The general solution is

$$y = c_1 + c_2 e^{-(b/a)t}$$

and this tends to c_1 as $t \to \infty$.
With our initial conditions, it follows that

$$c_1 + c_2 = y_0 \quad \text{and} \quad -\frac{b}{a}c_2 = y_0'$$

implies

$$c_1 = y_0 + \frac{a}{b}y_0' \quad \text{and} \quad c_2 = -\frac{a}{b}y_0'.$$

Additional Problem

1. An equation of the form

$$t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0, \qquad t > 0 \tag{1}$$

where α and β are real constants, is called an Euler equation.

(a) Let $x = \ln t$ and calculate dy/dt and d^2y/dt^2 in terms of dy/dx and d^2y/dx^2.

Solution: An exercise for the reader.

(b) Use the results of part (a) to transform Eq. (ii) into

$$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0. \tag{2}$$

Observe that Eq. (2), has constant coefficients. If $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of Eq. (2), then $y_1(\ln t)$ and $y_2(\ln t)$ form a fundamental set of solutions of Eq. (1).

Solution: An exercise for the reader.