Discussion #29 4/13/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Problems

1. (a) How many real eigenvalues, counting multiplicity, will a symmetric 100×100 matrix have?

   Solution: The matrix will have 100, since the matrix is symmetric.

   (b) Let A be an $n \times n$ symmetric matrix of rank r. Explain why the spectral decomposition of A represents A as the sum of r rank 1 matrices.

   Solution: This follows from the Spectral Value Decomposition Theorem. The each matrix is of the form

   $$\mathbf{v}_k \cdot \mathbf{v}_k^T$$

   and we have shown in a past worksheet these matrices have rank 1.

2. Consider the following passage from §7.4 of Lay:

   > "The *singular values* of A are the square roots of the eigenvalues of A^TA, denoted by $\sigma_1, \ldots, \sigma_n$, and they are arranged in decreasing order. That is, $\sigma_i = \sqrt{\lambda_i}$ for $1 \leq i \leq n$."

   Is an advantage gained by listing the singular values in a specific order?

   Solution: This makes the diagonal matrix in the SVD of a matrix unique, the diagonal entries in Σ can only be placed in one order.

   If a matrix has repeated singular values, then the choice of the corresponding column vectors in V and U can change. In other words, a matrix can have several different singular value decompositions, but Σ term will always remain the same.

3. Find the spectral decomposition of

   $$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix}$$

   given

   $$A = PDP^{-1}$$

   where

   $$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

Solution: We have

$$A = -4A_{-4} + 4A_4 + 7A_7$$

where

$$A_{-4} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

as well as

$$A_4 = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/6 & -1/3 & 1/6 \\ -1/3 & 2/3 & -1/3 \\ 1/6 & -1/3 & 1/6 \end{bmatrix}$$

and

$$A_7 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

4. Find the singular values of

$$A = \begin{bmatrix} 5 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$$

Solution: We have

$$A^T A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 1 \end{bmatrix}$$

so the eigenvalues for this diagonal matrix are 25 and 1. Thus A's singular values are 5 and 1.

5. Find the Singular Value Decomposition of

$$\begin{bmatrix} 4 & 6 \\ 0 & 4 \end{bmatrix}.$$

Solution: We have

$$A^T A = \begin{bmatrix} 4 & 0 \\ 6 & 4 \end{bmatrix} \cdot \begin{bmatrix} 4 & 6 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 16 & 24 \\ 24 & 52 \end{bmatrix}$$

then

$$\det(A^T A - \lambda I) = \begin{vmatrix} 16-\lambda & 24 \\ 24 & 52-\lambda \end{vmatrix} = (16-\lambda)(52-\lambda) - 24^2 = (\lambda - 64)(\lambda - 4)$$

so the singular values of A are 8 and 2, while the eigenvalues of A^TA are 64 and 4.

Case $\lambda = 64$:

Now

$$A^TA - 64I = \begin{bmatrix} -48 & 24 \\ 24 & -12 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$$

so pick

$$\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}.$$

Case $\lambda = 4$:

We have

$$A^TA - 4I = \begin{bmatrix} 12 & 24 \\ 24 & 48 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

so pick

$$\mathbf{v}_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}.$$

We define $\sigma_1 = 8$, $\sigma_2 = 2$,

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \begin{bmatrix} 4 & 6 \\ 0 & 4 \end{bmatrix} \cdot \frac{1}{8} \cdot \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \begin{bmatrix} 4 & 6 \\ 0 & 4 \end{bmatrix} \cdot \frac{1}{2} \cdot \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

therefore

$$A = U\Sigma V^T = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}.$$

6. Suppose A has Singular Value Decomposition $U\Sigma V^T$.

   (a) Find the Singular Value Decomposition for A^{-1}, provided A^{-1} exists.

   Solution: We have

   $$A^{-1} = V\Sigma^{-1}U^T.$$

   (b) Find the Singular Value Decomposition for A^T.

   Solution: We have

   $$A^T = \left(U\Sigma V^T\right)^T = V\Sigma^T U^T.$$