Discussion #28 4/10/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Problems

1. Let A be an $n \times n$ matrix. Answer the following *True* or *False*. Explain your reasoning, or give a counterexample.
   (a) If A is orthogonally diagonalizable, then A is symmetric.

   Solution: True: This follows by Theorem 2 in §7.1

   (b) If A is not symmetric, then A has at least one non-real eigenvalue.

   Solution: False: Consider
   $$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
   then the only eigenvalue is 1.

   (c) If A is symmetric with eigenvalue λ repeated 5 times then the eigenspace corresponding to λ has dimension 5.

   Solution: True: The matrix is diagonalizable, and each eigenspace has it dimension equal to the multiplicity of its corresponding eigenvalue.

2. Suppose that A is a 2×2 matrix with eigenvalues 0 and 1 and corresponding eigenvectors $(1, 3)$ and $(3, -1)$.
   (a) Is A symmetric?

   Solution: Yes: The two eigenvectors are orthogonal so A must be symmetric.

   (b) Find A and check your answer to part (a).

   Solution: We have
   $$\begin{aligned} A &= PDP^{-1} \\ &= \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & -1/10 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 3 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & -1/10 \end{bmatrix} \\ &= \begin{bmatrix} 9/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix} \end{aligned}$$
   and $A = A^T$.

3. Let
   $$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Find a matrix Q that orthogonally diagonalizes A, and determine $Q^{-1}AQ$.

Solution: We have

$$\begin{aligned}
\det(A-\lambda I) &= \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} \\
&= 0 - 0 + (-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} \\
&= \lambda((1-\lambda)^2 - 1) \\
&= \lambda(1 - 2\lambda + \lambda^2 - 1) \\
&= \lambda(-2\lambda + \lambda^2) \\
&= \lambda^2(\lambda - 2)
\end{aligned}$$

so the eigenvalues are 0 and 2.

Case $\lambda = 0$: We have

$$A - 0I = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and so

$$\mathbf{v} = \begin{bmatrix} -c_2 \\ c_2 \\ c_3 \end{bmatrix} = c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Our eigenvectors are spanned by

$$(-1, 1, 0) \quad \text{and} \quad (0, 0, 1).$$

Case $\lambda = 2$: We have

$$A - 2I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so

$$(1, 1, 0)$$

spans the eigenspace for the eigenvalue 2.

Let

$$Q = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

then

$$D = Q^{-1}AQ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

4. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

(a) What is $\text{rank}(A)$? What is $\dim \text{Nul}(A)$?

Solution: Here

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

tells us $\text{rank}(A) = 1$, the number of pivots, while $\text{nullity}(A) = 3$, the number of free variables.

(b) Why is 0 an eigenvalue of A, and what is the dimension of the eigenspace corresponding to 0?

Solution: The matrix A is not invertible and so 0 must be an eigenvalue of A.

(c) Does A have any other eigenvalues besides 0? Explain.

Solution: Yes

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$$

and so we found an eigenvectors with an eigenvalue of 4.

5. Let A be a 4×4 matrix.

(a) If the eigenvalues of A are $1, -2, 3, -3$, can you figure out $\det(A)$?

Solution: We have

$$\det(A) = 1 \cdot (-2) \cdot 3 \cdot (-3) = 18$$

because the determinant is the product of the eigenvalues.

(b) What if the eigenvalues are $-1, 1, 2$?

Solution: Depending on the multiplicity of the eigenvalue 2, we can only say that

$$|\det(A)| = 2 \quad \text{or} \quad |\det(A)| = 4.$$

(c) What if the eigenvalues are $-1, 0, 1$?

Solution: We know that

$$\det(A) = 0$$

since 0 the product of the eigenvalues is 0.

6. Prove that if there is an orthogonal matrix that diagonalizes A, then A is symmetric. (See Question 1a.)

Solution: We have

$$A = QDQ^{-1} = QDQ^T$$

since Q is orthogonal. Now

$$A^T = \left(QDQ^T\right)^T = \left(Q^T\right)^T D^T Q^T = QDQ^T = A$$

since $D = D^T$. Thus A is symmetric.

(Here we used $(ABC)^T = C^T B^T A^T$, as we have done in prior problems.)