Discussion #24 3/30/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Problems

1. What can you say about the least squares solution of $A\mathbf{x} = \mathbf{b}$ when $\mathbf{b}$ is orthogonal to the columns of A?

   Solution: If $\mathbf{b}$ is orthogonal to the columns of A, then the projection of $\mathbf{b}$ onto $\mathrm{Col}(A)$ is $\mathbf{0}$. The least-squares solution $\hat{\mathbf{x}}$ of $A\mathbf{x} = \mathbf{b}$ is $A\hat{\mathbf{x}} = \mathbf{0}$.

2. Answer the following *True* or *False*. Justify each answer.

   (a) If $\mathbf{b}$ is in the column space of A, then every solution of $A\mathbf{x} = \mathbf{b}$ is a least-squares solution.

   Solution: True: We cannot minimize any further. In other words, there is no need to do least squares because we already have a solution.

   (b) The least-squares solution of $A\mathbf{x} = \mathbf{b}$ is the point in the column space of A closest to $\mathbf{b}$.

   Solution: False: The least squares solution $\hat{\mathbf{x}}$ lies in the domain, not the column space of A (which is the range of A).

   (c) A least-squares solution of $A\mathbf{x} = \mathbf{b}$ is a list of weights that, when applied to the columns of A, produces the orthogonal projection of $\mathbf{b}$ onto $\mathrm{Col}(A)$.

   Solution: True: That is how we construct the solution.

   (d) If $\hat{\mathbf{x}}$ is a least-squares solution of $A\mathbf{x} = \mathbf{b}$, then $\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}$.

   Solution: False: This does not hold if A's columns are linearly dependent.

3. Find the least squares solution $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}.$$

   Solution: We have

$$A^TA = \begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}, \quad A^T\mathbf{b} = \begin{bmatrix} -24 \\ -2 \end{bmatrix}$$

   and so

$$\mathbf{x}_0 = (A^TA)^{-1}A^T\mathbf{b} = \frac{1}{56}\begin{bmatrix} 10 & -8 \\ -8 & 12 \end{bmatrix}\begin{bmatrix} -24 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}.$$

4. Let

$$A = \begin{bmatrix} 3 & 4 \\ -2 & 1 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 11 \\ -9 \\ 5 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

Find $A\mathbf{u}$ and $A\mathbf{v}$ and compare them to $\mathbf{b}$. Will $\mathbf{u}$ be a possible least squares solution of $A\mathbf{x} = \mathbf{b}$?

Solution: Notice

$$A\mathbf{u} = \begin{bmatrix} 11 \\ -11 \\ 11 \end{bmatrix}, \quad \mathbf{b} - A\mathbf{u} = \begin{bmatrix} 0 \\ 2 \\ -6 \end{bmatrix}$$

$$A\mathbf{v} = \begin{bmatrix} 7 \\ -12 \\ 7 \end{bmatrix}, \quad \mathbf{b} - A\mathbf{v} = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}.$$

Here $\|\mathbf{b} - A\mathbf{u}\| > \|\mathbf{b} - A\mathbf{v}\| = \sqrt{29}$, so $\mathbf{u}$ cannot be the least squares solution.

5. Use the factorization $A = QR$ to find the least-squares solution of $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}.$$

Solution: We have

$$R\hat{\mathbf{x}} = Q^T\mathbf{b}$$

where

$$Q^T\mathbf{b} = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -1/3 & 2/3 & -2/3 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 17/2 \\ 9/2 \end{bmatrix}.$$

We row reduce $\left[R \mid Q^T\mathbf{b}\right]$ and see

$$\left[\begin{array}{cc|c} 2 & 3 & 17/2 \\ 0 & 5 & 9/2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3/2 & 17/4 \\ 0 & 5 & 9/2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 29/10 \\ 0 & 1 & 9/10 \end{array}\right]$$

thus

$$\hat{\mathbf{x}} = \begin{bmatrix} 29/10 \\ 9/10 \end{bmatrix}.$$

6. Suppose $A \in \mathbf{R}^{m\times n}$ with n linearly independent column vectors and $\mathbf{b} \in \mathbf{R}^m$. Use the normal equations to find a formula for

$$\text{proj}_{\text{Col}(A)}(\mathbf{b}) = \mathbf{b}_{\text{c}}$$

Solution: We have

$$\mathbf{b}_c = A\mathbf{x}_0 = A(A^TA)^{-1}A^T\mathbf{b}.$$

Note A^{-1} need not exist, so we cannot reduce further.

7. Find the equation of the line $y = mx + b$ that best fits the points $(-1,-1)$, $(1,0)$, and $(2,4)$ in the least-squares sense by following these steps:

(a) Write down the (inconsistent) system of three equations in two unknowns for this problem.

Solution: We have

$$\begin{aligned} -m + b &= -1 \\ m + b &= 0 \\ 2m + b &= 4 \end{aligned}$$

(b) Rewrite the system as a matrix equation $A\mathbf{x} = \mathbf{y}$, where $\mathbf{x} = \begin{bmatrix} m \\ b \end{bmatrix}$. What are A and $\mathbf{y}$?

Solution: We have

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} m \\ b \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}.$$

(c) Find A^T and form the equation $A^TA\mathbf{x} = A^T\mathbf{y}$.

Solution: It follows that

$$A^TA = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

and

$$A^T\mathbf{y} = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$$

(d) Since the columns of A are linearly independent, A^TA is invertible. Find $(A^TA)^{-1}$ and use it to solve the equation in part (c).

Solution: We get

$$\left(A^TA\right)^{-1} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = \frac{1}{14}\begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}$$

and so

$$\mathbf{x} = \left(A^T A\right)^{-1} A^T \mathbf{y} = \frac{1}{14}\begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} 9 \\ 3 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 21 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 0 \end{bmatrix}$$

(e) What is the equation of the line that best fits the given points?

Solution: The line of best fit is

$$y = \frac{3}{2}x.$$

Midterm review

Let's recall some of the things we did before spring break.

1. Let $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$.
   (a) Find the eigenvalues of A and a basis for each eigenspace.
   (b) Diagonalize A.