Discussion #23 3/20/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Problems

1. What is the definition of a projection matrix?

   Solution: We have $P^2 = P$.

2. Create an orthogonal projection matrix. Can you create a projection matrix that is not an orthogonal projection matrix?

   Solution: Let
   $$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
   the P represents the orthogonal projection onto the x-axis. We can have
   $$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$
   which satisfies
   $$A^2 = A$$
   and also projects onto the x-axis but it is not orthogonal! Just draw a picture of $A\mathbf{e}_1$ and $A\mathbf{e}_2$ to convince yourself.

3. Answer the following *True* or *False*. Justify each answer.

   (a) If W is a subspace of $\mathbf{R}^n$ and if $\mathbf{v}$ is in both W and $W^\perp$, then $\mathbf{v}$ must be the zero vector.

   Solution: True: the only vector that is orthogonal to both W and $W^\perp$ is the zero vector.

   (b) In the Orthogonal Decomposition Theorem, each term in
   $$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p}\mathbf{u}_p$$
   for $\hat{\mathbf{y}}$ is itself an orthogonal projection of $\mathbf{y}$ onto a subspace of W.

   Solution: True: We have
   $$\frac{\mathbf{y} \cdot \mathbf{u}_k}{\mathbf{u}_k \cdot \mathbf{u}_k}\mathbf{u}_k$$
   is an orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}_k$.

   (c) If $\mathbf{y} = \mathbf{z}_1 + \mathbf{z}_2$, where $\mathbf{z}_1$ is in a subspace W and $\mathbf{z}_2$ is in $W^\perp$, then $\mathbf{z}_1$ must be the orthogonal projection of $\mathbf{y}$ onto W.

Solution: **True**: By §6.3 Theorem 8, the decomposition is unique.

(d) The best approximation to $\mathbf{y}$ by elements of a subspace W is given by the vector $\mathbf{y} - \text{proj}_W\, \mathbf{y}$.

Solution: False: The best approximation is $proj_W\mathbf{y}$.

(e) If an $n \times p$ matrix U has orthonormal columns, then $UU^T\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbf{R}^n$.

Solution: False: We need not have $\mathbf{x} \in \text{Col}(U)$. For matrices that are not square, in particular have more rows than columns, we cannot assume $\text{Col}(U) = \mathbf{R}^n$.

4. Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are orthogonal.

   Does it follow that $\mathbf{v}_1$, $\mathbf{v}_2$, and $c\mathbf{v}_3$ are orthogonal?

   Solution: If $c \neq 0$, the vectors are still orthogonal. If $c = 0$, then the vectors are still orthogonal, because an orthogonal set can still include the zero vector.

5. Suppose $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w} = 0$. Does it follow that $\mathbf{u} \cdot \mathbf{w} = 0$?

   Solution: No, assume $\mathbf{u} \neq \mathbf{0}$,

$$\mathbf{u} \cdot \mathbf{v} = 0 \quad \text{and} \quad \mathbf{w} = -\mathbf{u}.$$

   We have

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot (-\mathbf{u}) = 0.$$

   However

$$\mathbf{u} \cdot -\mathbf{u} = 0 \quad \text{if and only if} \quad \mathbf{u} = \mathbf{0}.$$

6. Find the closest point to $\mathbf{v}$ in the subspace W where

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad W = \text{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

   Solution: We have

$$\text{proj}_W(\mathbf{v}) = \left(-\frac{1}{2}\right) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{6}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 5/2 \\ 2 \end{bmatrix}$$

7. Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Note that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal but that $\mathbf{u}_3$ is not orthogonal to $\mathbf{u}_1$ or $\mathbf{u}_2$. It can be shown that $\mathbf{u}_3$ is not in the subspace W spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$. Use this fact to construct a nonzero vector $\mathbf{v}$ in $\mathbf{R}^3$ that is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$.

Solution: We have options:

1. Use the cross-product of $\mathbf{u}_1$ and $\mathbf{u}_2$.
2. Apply Gram-Schmidt.

Cross Product:

$$\begin{aligned}
\mathbf{u}_1 \times \mathbf{u}_2 &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 1 & -2 \\ 5 & -1 & 2 \end{vmatrix} \\
&= \mathbf{e}_1 \begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} - \mathbf{e}_2 \begin{vmatrix} 1 & -2 \\ 5 & 2 \end{vmatrix} + \mathbf{e}_3 \begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} \\
&= \mathbf{e}_1 (1 \cdot 2 - (-1) \cdot (-2)) - \mathbf{e}_2 (1 \cdot 2 - 5 \cdot (-2)) + \mathbf{e}_3 (1 \cdot (-1) - 5 \cdot 1) \\
&= \mathbf{e}_1 (2 - 2) - \mathbf{e}_2 (2 + 10) + \mathbf{e}_3 (-1 - 5) \\
&= 0\mathbf{e}_1 - 12\mathbf{e}_2 - 6\mathbf{e}_3 \\
&= \begin{bmatrix} 0 \\ -12 \\ -6 \end{bmatrix}
\end{aligned}$$

Gram-Schmidt: Here

$$\begin{aligned}
b_{3,1} &= \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} = \frac{0 \cdot 1 + 0 \cdot 1 + 1 \cdot (-2)}{1^2 + 1^2 + (-2)^2} = \frac{-2}{6} = -\frac{1}{3} \\
b_{3,2} &= \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} = \frac{0 \cdot 5 + 0 \cdot (-1) + 1 \cdot 2}{5^2 + (-1)^2 + 2^2} = \frac{2}{30} = \frac{1}{15}
\end{aligned}$$

and

$$\mathbf{v}_3 = \mathbf{x}_3 - b_{3,1} \cdot \mathbf{v}_1 - b_{3,2} \cdot \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} - \frac{1}{15} \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/5 \\ 1/5 \end{bmatrix}$$

8. Let

$$\mathbf{y} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$$

and $W = \text{Span}\{\mathbf{u}_1\}$.

(a) Let U be the 2×1 matrix whose only column is $\mathbf{u}_1$. Compute $U^T U$ and UU^T.

Solution: We have

$$UU^T = \begin{bmatrix} \frac{\sqrt{10}}{10} \\ -\frac{3\sqrt{10}}{10} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{10}}{10} & -\frac{3\sqrt{10}}{10} \end{bmatrix} = \begin{bmatrix} 1/10 & -3/10 \\ -3/10 & 9/10 \end{bmatrix}$$

and

$$U^TU = \begin{bmatrix} \frac{\sqrt{10}}{10} & -\frac{3\sqrt{10}}{10} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{10}}{10} \\ -\frac{3\sqrt{10}}{10} \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

(b) Compute $\text{proj}_W(\mathbf{y})$ and $(UU^T)\mathbf{y}$.

Solution: Here

$$\begin{aligned} \text{proj}_W(\mathbf{y}) &= \langle \mathbf{y}, \mathbf{u}_1 \rangle \, \mathbf{u}_1 \\ &= \left(7 \cdot 1/\sqrt{10} + 9 \cdot \left(-3/\sqrt{10}\right)\right) \mathbf{u}_1 \\ &= -\frac{20}{\sqrt{10}} \mathbf{u}_1 \\ &= -2\sqrt{10} \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 6 \end{bmatrix} \end{aligned}$$

and

$$UU^T\mathbf{y} = \begin{bmatrix} 1/10 & -3/10 \\ -3/10 & 9/10 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 9 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

9. Let W be a subspace of $\mathbf{R}^n$. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbf{R}^n$ and let $\mathbf{z} = \mathbf{x} + \mathbf{y}$. If $\mathbf{u}$ is the projection of $\mathbf{x}$ onto W and $\mathbf{v}$ is the projection of $\mathbf{y}$ onto W, show that $\mathbf{u} + \mathbf{v}$ is the projection of $\mathbf{z}$ onto W.

Solution: Let U be the matrix whose columns form an orthonormal basis for W. Then

$$\begin{aligned} \text{proj}_W(\mathbf{z}) &= UU^T\mathbf{z} \\ &= UU^T(\mathbf{x} + \mathbf{y}) \\ &= UU^T\mathbf{x} + UU^T\mathbf{y} \\ &= \text{proj}_W(\mathbf{x}) + \text{proj}_W(\mathbf{y}) \\ &= \mathbf{u} + \mathbf{v}. \end{aligned}$$

10. Answer the following *True* or *False*. Justify each answer.

(a) If $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ with $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ linearly independent, and if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set in W, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for W.

Solution: False: We require the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to be nonzero.

(b) If $\mathbf{x}$ is not in a subspace W, then $\mathbf{x} - \text{proj}_W(\mathbf{x})$ is not zero.

Solution: True: If $\mathbf{x} \notin W$, then $\mathbf{x} \neq \text{proj}_W(\mathbf{x})$, because $\text{proj}_W(\mathbf{x})$ is in W.

(c) In a QR factorization, say $A = QR$ (when A has linearly independent columns), the columns of Q form an orthonormal basis for the column space of A.

Solution: True: This is Theorem 12 from §6.4.

11. Use the Gram-Schmidt process to transform the basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ into an orthonormal basis.

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Solution: Let

$$\mathbf{v}_1 = \mathbf{x}_1 \quad \text{and} \quad \mathbf{v}_2 = \mathbf{x}_2$$

since the two vectors are already orthogonal. Now

$$b_{3,1} = \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} = \frac{1 \cdot 1 + 2 \cdot 1 + 1 \cdot 1}{1^2 + 1^2 + 1^2} = \frac{4}{3}$$
$$b_{3,2} = \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} = \frac{1 \cdot (-1) + 2 \cdot 1 + 1 \cdot 0}{(-1)^2 + 1^2 + 0^2} = \frac{1}{2}$$

and

$$\mathbf{v}_3 = \mathbf{x}_3 - b_{3,1} \cdot \mathbf{v}_1 - b_{3,2} \cdot \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \frac{4}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \\ -1/3 \end{bmatrix}$$

12. Find the QR factorization for

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

given

$$Q = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}.$$

Solution: We have

$$\begin{aligned} R &= Q^T A \\ &= \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}. \end{aligned}$$

13. Find the QR factorization for

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}.$$

Hint: The column vectors appeared in the lecture on §6.4 - The Gram-Schmidt Process.

Solution: First,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

are linearly independent. Let

$$\mathbf{w}_1 = \mathbf{v}_1$$

then

$$\mathbf{w}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

and

$$\mathbf{w}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - \frac{4}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 1/3 \\ 2/3 \end{bmatrix}.$$

This tells us

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix}$$

where the columns of Q are the normalized vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. We have

$$\begin{aligned} R &= Q^T A \\ &= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{2} & \sqrt{2} & 2\sqrt{2} \\ 0 & \sqrt{3} & 1/\sqrt{3} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix} \end{aligned}$$

and if we rationalize the denominators

$$Q = \begin{bmatrix} \sqrt{2}/2 & \sqrt{3}/3 & -\sqrt{6}/6 \\ \sqrt{2}/2 & -\sqrt{3}/3 & \sqrt{6}/6 \\ 0 & \sqrt{3}/3 & \sqrt{6}/3 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} \sqrt{2} & \sqrt{2} & 2\sqrt{2} \\ 0 & \sqrt{3} & \sqrt{3}/3 \\ 0 & 0 & \sqrt{6}/3 \end{bmatrix}.$$

14. Suppose $A = QR$, where $Q \in \mathbf{R}^{m \times n}$ and $R \in \mathbf{R}^{n \times n}$.

If A has linearly independent column vectors, prove that R must be invertible.

Note that A need not be square.

Solution: Suppose $R\mathbf{x} = \mathbf{0}$, we want to show $\mathbf{x} = \mathbf{0}$.

Then

$$QR\mathbf{x} = Q \cdot \mathbf{0} = A\mathbf{x} = \mathbf{0}.$$

But Q is one-to-one and A is one-to-one since Q^{-1} exists and the columns of A are linearly independent. Thus $\mathbf{x} = \mathbf{0}$ and R has linearly independent columns.