Discussion #21 3/16/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Problems

1. Answer the following *True* or *False*. Explain your reasoning, or give a counterexample.

   (a) If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbf{R}^4$ such that $\mathbf{x} \cdot \mathbf{y} = 0$, then either $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$.

   Solution: False: Let
   $$\mathbf{x} = \mathbf{e}_1 \quad \text{and} \quad \mathbf{y} = \mathbf{e}_2.$$

   (b) If $\mathbf{u} \neq \mathbf{0}$ is a nonzero vector in $\mathbf{R}^n$ then the vector $\frac{\mathbf{u}}{\|\mathbf{u}\|}$ has norm 1.

   Solution: True: We have
   $$\left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = \frac{1}{\|\mathbf{u}\|} \cdot \|\mathbf{u}\| = 1$$
   for $\mathbf{u} \neq \mathbf{0}$.

   (c) If A is a 3×3 matrix, and $\mathbf{x}$ and $\mathbf{y}$ are column vectors in $\mathbf{R}^3$, then
   $$A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A\mathbf{y}.$$

   Solution: False: Let
   $$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$
   then
   $$A\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad A\mathbf{y} = \begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix}$$
   and
   $$A\mathbf{x} \cdot \mathbf{y} = 10 \quad \text{and} \quad \mathbf{x} \cdot A\mathbf{y} = 14.$$
   What does hold true is
   $$A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A^T\mathbf{y}. \tag{1}$$

   (d) If A is an $m \times n$ matrix, the equation $A\mathbf{x} = \mathbf{y}$ has a solution if and only if $\mathbf{y}$ is in the column space of A.

   Solution: True: We proved this in a prior discussion.

   (e) If A is an $m \times n$ matrix whose columns are linearly independent, then A^TA is invertible.

Solution: This follows by a direct computation. Suppose

$$A\mathbf{x} = \mathbf{0}$$

then

$$A^T A\mathbf{x} = A^T\mathbf{0} = \mathbf{0}.$$

Now suppose

$$A^T A\mathbf{x} = \mathbf{0}$$

then

$$\mathbf{x} \cdot (A^T A\mathbf{x}) = 0$$

because we assume $A^T A\mathbf{x}$ is the zero vector. Then by equation (1) applied to $A^T A\mathbf{x}$ and the vector $\mathbf{x}$ we have

$$0 = \mathbf{x} \cdot (A^T A\mathbf{x}) = \mathbf{x}^T A^T A\mathbf{x} = (A\mathbf{x})^T (A\mathbf{x}) = A\mathbf{x} \cdot A\mathbf{x} = \|A\mathbf{x}\|^2$$

and thus $A\mathbf{x} = \mathbf{0}$.

This shows $\text{Nul}(A) = \text{Nul}(A^T A)$ and so if A's column vectors are linearly independent it follows that $A^T A$ and A's null space is just the zero vector. This means $\text{nullity}(A^T A) = 0$ and so $A^T A$ has full rank. With $A^T A$ square it follows that $A^T A$ has linearly independent column vectors.

(f) Any basis for $\mathbf{P}_n$ (the vector space of all polynomials of degree $\leq n$) must contain a polynomial of degree k for each $k = 0, 1, 2, \ldots, n$.

Solution: False: The set

$$\mathcal{B} = \{1 - t, 1 + t\}$$

is a basis for $\mathbf{P}_1$, just as

$$\{(1, 1), (1, -1)\}$$

is a basis for $\mathbf{R}^2$.

(g) Let $\mathcal{B} = \{\mathbf{u}_1, \ldots \mathbf{u}_n\}$ be an orthogonal set in $\mathbf{R}^m$. Then

$$U = \begin{bmatrix} \mathbf{u}_1 & \ldots & \mathbf{u}_p \end{bmatrix}$$

satisfies

$$U^T U = I_n.$$

Solution: False: Consider

$$\mathcal{B} = \{(1, 1), (1, -1)\}$$

then

$$U = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

satisfies

$$U^T U = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \neq I_2.$$

If $\mathcal{B}$ is an orthonormal set, then the result would hold.

(h) If A is a 4×4 matrix whose column vectors form an orthonormal basis for $\mathbf{R}^4$, then A is invertible.

Solution: **True**: If vectors form a basis, regardless of whether they are orthogonal, they must be linearly independent. Of course, a collection of orthogonal vectors is always linearly independent.

Either way, A is invertible because its column vectors are linearly independent.

2. If $\mathbf{a}, \mathbf{b} \in \mathbf{R}^n$, what is the difference between

$$\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \quad \text{and} \quad \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a}?$$

Solution: We have

$$\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \in \mathbf{R}$$

while

$$\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a} \in \mathbf{R}^n.$$

3. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}.$$

(a) Compute $\|\mathbf{u} + \mathbf{v}\|$ and $\|\mathbf{u}\| + \|\mathbf{v}\|$. Which is larger?

Solution: We have

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

while

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\| &= \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6} \\ \|\mathbf{u}\| &= \sqrt{1^2 + 0^2 + (-2)^2} = \sqrt{5} \\ \|\mathbf{v}\| &= \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} \\ \|\mathbf{u}\| + \|\mathbf{v}\| &= \sqrt{5} + \sqrt{3}. \end{aligned}$$

Since $\sqrt{6} < \sqrt{5} + \sqrt{3}$, we conclude

$$\|\mathbf{u} + \mathbf{v}\| < \|\mathbf{u}\| + \|\mathbf{v}\|.$$

(b) Compute $\|\mathbf{v} \cdot \mathbf{w}\|$ and $\|\mathbf{v}\| \, \|\mathbf{w}\|$. Which is larger?

Solution: Notice

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{w} &= 1 \cdot 0 + 1 \cdot 2 + 1 \cdot (-1) = 1 \\
\|\mathbf{v} \cdot \mathbf{w}\| &= |1| = 1 \\
\|\mathbf{w}\| &= \sqrt{0^2 + 2^2 + (-1)^2} = \sqrt{5} \\
\|\mathbf{v}\| \cdot \|\mathbf{w}\| &= \sqrt{3} \cdot \sqrt{5} = \sqrt{15}
\end{aligned}$$

and therefore

$$\|\mathbf{v} \cdot \mathbf{w}\| < \|\mathbf{v}\| \cdot \|\mathbf{w}\|.$$

(c) Compute $\mathbf{u}^T\mathbf{v}$, $\mathbf{v}^T\mathbf{u}$, and $\mathbf{u} \cdot \mathbf{v}$. Is $\mathbf{v}\mathbf{u}^T$ defined? If so, compute it. If not, explain.

Solution: We find

$$\begin{aligned}
\mathbf{u}^T\mathbf{v} &= [1 \quad 0 \quad -2] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot 1 + 0 \cdot 1 + (-2) \cdot 1 = -1 \\
\mathbf{v}^T\mathbf{u} &= [1 \quad 1 \quad 1] \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = 1 \cdot 1 + 1 \cdot 0 + 1 \cdot (-2) = -1 \\
\mathbf{u} \cdot \mathbf{v} &= 1 \cdot 1 + 0 \cdot 1 + (-2) \cdot 1 = -1
\end{aligned}$$

so each term is the same. Now consider $\mathbf{v}\mathbf{u}^T$

$$\mathbf{v}\mathbf{u}^T = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 0 & -2 \\ 1 & 0 & -2 \end{bmatrix}.$$

4. Let W be a subspace of $\mathbf{R}^n$. Define the **orthogonal complement** of W to be the subspace

$$W^\perp = \{\mathbf{v} \in \mathbf{R}^n : \mathbf{w} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{w} \in W\}.$$

(a) Consider the subspace $W = \operatorname{Span}\{(1,1,1), (2,0,-1)\}$ of $\mathbf{R}^3$. Find a vector which spans $W^\perp$.

Solution: Choose

$$\begin{aligned}
(1,1,1)\times(2,0,-1) &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{vmatrix} \\
&= \mathbf{e}_1 \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} - \mathbf{e}_2 \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + \mathbf{e}_3 \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} \\
&= \mathbf{e}_1(1\cdot(-1) - 1\cdot 0) - \mathbf{e}_2(1\cdot(-1) - 1\cdot 2) + \mathbf{e}_3(1\cdot 0 - 1\cdot 2) \\
&= \mathbf{e}_1(-1) - \mathbf{e}_2(-3) + \mathbf{e}_3(-2) \\
&= (-1, 3, -2).
\end{aligned}$$

(b) Express the vector $(2, 1, -3)$ in the form $\mathbf{w} + \mathbf{w}^{\perp}$, where $\mathbf{w} \in W$ and $\mathbf{w}^{\perp} \in W^{\perp}$.

Solution: For now, we will use the standard approach. We will see later in chapter 6 how to do this in a better manner.

Let

$$\mathbf{v}_1 = (1,1,1), \quad \mathbf{v}_2 = (2,0,-1), \quad \text{and} \quad \mathbf{v}_3 = (-1,3,-2)$$

then we solve

$$[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \mid \mathbf{v}]$$

$$\begin{aligned}
\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 1 & 0 & 3 & 1 \\ 1 & -1 & -2 & -3 \end{array}\right] &\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -2 & 4 & -1 \\ 0 & -3 & -1 & -5 \end{array}\right] \\
&\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & 1/2 \\ 0 & 0 & -7 & -7/2 \end{array}\right] \\
&\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & 1/2 \end{array}\right]
\end{aligned}$$

tells us

$$\mathbf{v} = -1/2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3/2 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + 1/2 \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

so pick

$$\mathbf{w} = -1/2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3/2 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -1/2 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{w}^{\perp} = \begin{bmatrix} -1/2 \\ 3/2 \\ -1 \end{bmatrix}.$$

5. Let

$$W = \text{Span}\{(1,2,3)\}.$$

Find a basis for $W^\perp$.

Solution: We can think of $W^\perp$ as the plane with normal vector $(1,2,3)$. Then

$$x + 2y + 3z = 0$$

is the only plane with the given normal vector that passes through the origin. We can then find two linearly independent vectors that lie in the plane. Let $x = 2$ and $z = 0$ then we get

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

and if $x = 0$ and $y = 3$ we have

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}.$$

Thus

$$W^\perp = \text{Span}\{(2,-1,0),(0,3,-2)\}.$$

6. Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has the property that

$$\|A\mathbf{x}\| = \|\mathbf{x}\|$$

for all $\mathbf{x} \in \mathbf{R}^2$.

(a) Construct 3 different matrices that have the same properties as A. What do your 3 matrices all have in common?

Solution: We can have

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad A_3 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2}. \end{bmatrix}$$

In each case we have

$$\|A\mathbf{e}_1\| = \|\mathbf{e}_1\| \quad \text{and} \quad \|A\mathbf{e}_2\| = \|\mathbf{e}_2\|$$

so in general

$$\|A\mathbf{e}_1\| = \left\| \begin{bmatrix} a \\ c \end{bmatrix} \right\| = \sqrt{a^2 + c^2} = 1$$

and

$$\|A\mathbf{e}_2\| = \left\| \begin{bmatrix} b \\ d \end{bmatrix} \right\| = \sqrt{b^2 + d^2} = 1$$

since $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors. This means that

$$a^2 + b^2 + c^2 + d^2 = 2.$$

So an interesting question would be, if

$$a^2 + b^2 + c^2 + d^2 = 2$$

does it follow that

$$\|A\mathbf{x}\| = \|\mathbf{x}\|$$

for all $\mathbf{x}$?

(b) Compute $\|A^2\mathbf{x}\|$.

Solution: We have

$$\|A^2\mathbf{x}\| = \|A(A\mathbf{x})\| = \|A\mathbf{x}\| = \|\mathbf{x}\|.$$

(c) What are the possible eigenvalues for A?

Solution: If we have an eigenvector $\mathbf{v}$ with eigenvector λ it follows that

$$\|\mathbf{v}\| = \|A\mathbf{v}\| = \|\lambda\mathbf{v}\| = |\lambda|\|\mathbf{v}\|$$

and so

$$|\lambda| = 1.$$

(d) Can A be symmetric?

Solution: Yes:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(e) Must A be symmetric?

Solution: No:

$$A_3 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2}. \end{bmatrix}.$$

(f) If A^{-1} exists, compute $\|A^{-1}\mathbf{x}\|$.

Solution: If

$$\|A\mathbf{x}\| = \|\mathbf{x}\|$$

and $\mathbf{b} = A^{-1}\mathbf{x}$ then it must follow that

$$\|A^{-1}\mathbf{x}\| = \|\mathbf{b}\| = \|A(A^{-1}\mathbf{x})\| = \|(AA^{-1})\mathbf{x}| = \|I\mathbf{x}\| = \|\mathbf{x}\|.$$

(g) Must A be invertible?

Solution: **Yes**: Suppose

$$A\mathbf{x} = \mathbf{0}$$

then

$$\|\mathbf{x}\| = \|A\mathbf{x}\| = 0$$

and so $\mathbf{x}$ must be the zero vector. Hence A is invertible.

7. (a) Let θ be a real number. Show that the vectors $\mathbf{v}_1 = (\cos(\theta), \sin(\theta))$ and $\mathbf{v}_2 = (-\sin(\theta), \cos(\theta))$ form an orthonormal basis for $\mathbf{R}^2$.

Solution: It follows that

$$R = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

satisfies

$$\det(R) = \cos^2(\theta) + \sin^2(\theta) = 1 \neq 0.$$

This shows the vectors are linearly independent (This avoids us having to prove the vectors are nonzero, because an orthogonal/orthonormal set that includes the zero vector is linearly dependent). Now

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$$

so the vectors are orthogonal. While

$$\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \cos^2(\theta) + \sin^2(\theta) = 1$$

makes them orthonormal.

(b) Let $\mathbf{u}_1 = (1, 1)$ and $\mathbf{u}_2 = (0, -1)$. Find a 2×2 matrix A which induces a linear transformation T_A such that $T_A(\mathbf{u}_1) = \mathbf{v}_1$ and $T_A(\mathbf{u}_2) = \mathbf{v}_2$.

Solution: Let

$$B = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

We seek a matrix A such that

$$AB = R$$

because

$$A\mathbf{u}_1 = \mathbf{v}_1 \quad \text{and} \quad A\mathbf{u}_2 = \mathbf{v}_2.$$

Since U is invertible,

$$A = RB^{-1}.$$

where

$$B^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = B,$$

and hence

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) - \sin(\theta) & \sin(\theta) \\ \sin(\theta) + \cos(\theta) & -\cos(\theta) \end{bmatrix}.$$

Observe,

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} = \mathbf{v}_1, \quad A \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} = \mathbf{v}_2$$

and so the transformation is defined correctly.

8. (a) Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis for $\mathbf{R}^3$, where $\mathbf{v}_1 = \left(1/\sqrt{2}, 1/\sqrt{2}, 0\right)$, $\mathbf{v}_2 = \left(-1/\sqrt{2}, 1/\sqrt{2}, 0\right)$, and $\mathbf{v}_3 = (0, 0, 1)$.

Solution: It follows immediately that

$$\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

and so the vectors are orthonormal.

(b) Find the coordinates of $\mathbf{w} = \left(\sqrt{2}, 3\sqrt{2}, -4\right)$ with respect to this basis.

Solution: We could solve $A\mathbf{x} = \mathbf{w}$ where

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} \sqrt{2} \\ 3\sqrt{2} \\ -4 \end{bmatrix}$$

but things are easier because our basis is orthonormal. We have

$$\mathbf{w} = (\mathbf{w} \cdot \mathbf{v}_1)\mathbf{v}_1 + (\mathbf{w} \cdot \mathbf{v}_2)\mathbf{v}_2 + (\mathbf{w} \cdot \mathbf{v}_3)\mathbf{v}_3$$

so

$$\begin{aligned} \mathbf{w} \cdot \mathbf{v}_1 &= \sqrt{2} \cdot 1/\sqrt{2} + 3\sqrt{2} \cdot 1/\sqrt{2} + (-4) \cdot 0 = 1 + 3 = 4 \\ \mathbf{w} \cdot \mathbf{v}_2 &= \sqrt{2} \cdot \left(-1/\sqrt{2}\right) + 3\sqrt{2} \cdot 1/\sqrt{2} + (-4) \cdot 0 = -1 + 3 = 2 \\ \mathbf{w} \cdot \mathbf{v}_3 &= \sqrt{2} \cdot 0 + 3\sqrt{2} \cdot 0 + (-4) \cdot 1 = -4 \end{aligned}$$

tells us

$$\mathbf{x} = 4 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} - 4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This is equivalent to computing

$$\mathbf{x} = A^T\mathbf{w}$$

by direct matrix/vector multiplication.

(c) Let

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Show that $A\mathbf{v}_1 = 2\mathbf{v}_1$, $A\mathbf{v}_2 = 4\mathbf{v}_2$, and $A\mathbf{v}_3 = -\mathbf{v}_3$.

Solution: We have

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} \\ 2/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -4/\sqrt{2} \\ 4/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

and so we have an orthonormal basis of eigenvectors.

(d) Compute $A\mathbf{w}$ directly, and again by using your work from parts (b) and (c).

Solution: We have

$$\begin{aligned} A\mathbf{w} &= A(4\mathbf{v}_1 + 2\mathbf{v}_2 - 4\mathbf{v}_3) \\ &= 4A\mathbf{v}_1 + 2A\mathbf{v}_2 - 4A\mathbf{v}_3 \\ &= 8\mathbf{v}_1 + 8\mathbf{v}_2 + 4\mathbf{v}_3 \\ &= \begin{bmatrix} 0 \\ 8\sqrt{2} \\ 4 \end{bmatrix} \end{aligned}$$

while a direct matrix/vector multiplication computation shows

$$A\mathbf{w} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{2} \\ 3\sqrt{2} \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 8\sqrt{2} \\ 4 \end{bmatrix}$$

9. A matrix is **orthogonal** if it is square and its columns are orthonormal.

Let A be a 2×2 orthogonal matrix.

(a) Show that A is of the form

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix},$$

for some value of θ in the interval $0 \leq \theta < 2\pi$.

Solution: We know that any two nonzero vectors in $\mathbf{R}^2$ are orthogonal if

$$\mathbf{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -b \\ a \end{bmatrix}$$

or

$$\mathbf{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} b \\ -a \end{bmatrix}$$

and if the vectors are unit vectors they can be identified in polar form as

$$\mathbf{v}_1 = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}.$$

Thus our matrix has the form

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}.$$

(b) Geometrically, what does an orthonormal basis for $\mathbf{R}^2$ look like?

Solution: A rotation of the xy-plane.

(c) What does an orthonormal basis for $\mathbf{R}^3$ look like?

Solution: A rigid transformation of the x, y, and z axes.