Discussion #20 3/13/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Problems

1. Answer the following *True* or *False*. Explain your reasoning, or give a counterexample.

   (a) The vectors

   $$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1+2i \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 6i \\ -6-3i \end{bmatrix}$$

   are linearly independent in $\mathbf{C}^2$.

   Solution: False: We have

   $$3i\mathbf{v}_1 = \begin{bmatrix} 6i \\ -6-3i \end{bmatrix} = \mathbf{v}_2$$

   so

   $$3i\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}$$

   is a dependency relation.

   (b) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation that rotates the plane by θ radians counterclockwise. Then T's matrix representation is **not** diagonalizable.

   Solution: False: Let $\theta = k\pi$ where $k \in \mathbf{Z}$. Then for k-odd we have

   $$T(\mathbf{e}_1) = -\mathbf{e}_1 \quad \text{and} \quad T(\mathbf{e}_2) = -\mathbf{e}_2,$$

   so we have two linearly independent eigenvectors.

   (c) If $B = P^{-1}AP$ and $\mathbf{x}$ is an eigenvector of A corresponding to an eigenvalue λ, then $P^{-1}\mathbf{x}$ is an eigenvector of B corresponding also to λ.

   Solution: True: Since $\mathbf{x}$ is an eigenvector of A, we have

   $$A\mathbf{x} = \lambda\mathbf{x}.$$

   Multiplying both sides on the left by P^{-1} gives

   $$P^{-1}A\mathbf{x} = \lambda P^{-1}\mathbf{x}.$$

   Using $B = P^{-1}AP$, we can write

   $$B(P^{-1}\mathbf{x}) = (P^{-1}AP)(P^{-1}\mathbf{x}) = P^{-1}A\mathbf{x} = \lambda P^{-1}\mathbf{x}.$$

   Therefore $P^{-1}\mathbf{x}$ is an eigenvector of B corresponding to the same eigenvalue λ.

   (d) Let A be a complex (or real) $n \times n$ matrix, and let $\mathbf{x} \in \mathbf{C}^n$ be an eigenvector corresponding to an eigenvalue $\lambda \in \mathbf{C}$.

   Then for each nonzero complex scalar μ, the vector $\mu\mathbf{x}$ is an eigenvector of A.

Solution: **True**: Since $\mathbf{x}$ is an eigenvector of A corresponding to λ, we have

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Then for any nonzero complex scalar μ,

$$A(\mu\mathbf{x}) = \mu(A\mathbf{x}) = \mu(\lambda\mathbf{x}) = \lambda(\mu\mathbf{x}).$$

Thus $\mu\mathbf{x}$ is also an eigenvector of A corresponding to the same eigenvalue λ.

2. Let $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be bases for vector spaces V and W, respectively. Let $T : V \to W$ be a linear transformation with the property that

$$T(\mathbf{d}_1) = 2\mathbf{b}_1 - 3\mathbf{b}_2, \quad T(\mathbf{d}_2) = -4\mathbf{b}_1 + 5\mathbf{b}_2.$$

Find the matrix for T relative to $\mathcal{D}$ and $\mathcal{B}$.

Solution: We know

$$[T(\mathbf{d}_1)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \quad \text{and} \quad [T(\mathbf{d}_2)]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$$

Then the matrix for T relative to $\mathcal{D}$ and $\mathcal{B}$ is

$$\big[[T(\mathbf{d}_1)]_{\mathcal{B}} \quad [T(\mathbf{d}_2)]_{\mathcal{B}}\big] = \begin{bmatrix} 2 & -4 \\ -3 & 5 \end{bmatrix}.$$

3. Find $T(a_0 + a_1t + a_2t^2)$, if T is the linear transformation from $\mathbf{P}_2$ to $\mathbf{P}_2$ whose matrix relative to $\mathcal{B} = \{1, t, t^2\}$ is

$$[T]_{\mathcal{B}} = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}.$$

Solution: Let

$$[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}.$$

Then

$$[T]_{\mathcal{B}}[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3a_0 + 4a_1 \\ 5a_1 - a_2 \\ a_0 - 2a_1 + 7a_2 \end{bmatrix}.$$

Hence

$$T(a_0 + a_1t + a_2t^2) = (3a_0 + 4a_1) + (5a_1 - a_2)t + (a_0 - 2a_1 + 7a_2)t^2.$$

4. Find the complex-valued eigenvalues of

$$A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$$

and a basis for each eigenspace in $\mathbf{C}^2$.

Solution: The characteristic polynomial is

$$\chi_A(\lambda) = \begin{vmatrix} 5-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = (5-\lambda)(3-\lambda) - (-2)\cdot 1 = \lambda^2 - 8\lambda + 17.$$

Hence

$$\lambda = \frac{8 \pm \sqrt{64-68}}{2} = \frac{8 \pm 2i}{2} = 4 \pm i.$$

For $\lambda = 4 + i$,

$$A - (4+i)I = \begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix} \sim \begin{bmatrix} 1 & -1-i \\ 0 & 0 \end{bmatrix}$$

Then

$$\mathbf{v}_1 = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}.$$

For $\lambda = 4 - i$, we take

$$\mathbf{v}_2 = \overline{\mathbf{v}_1} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

Thus,

$$E_{4+i} = \text{Span}\left\{ \begin{bmatrix} 1+i \\ 1 \end{bmatrix} \right\}, \quad E_{4-i} = \text{Span}\left\{ \begin{bmatrix} 1-i \\ 1 \end{bmatrix} \right\}.$$

5. Determine if

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

are similar.

Solution: They are similar. Notice the two matrices have the same eigenvalues, and are both diagonal matrices. We can choose

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

whose columns are linearly independent eigenvectors of A, and see that

$$PAP^{-1} = PAP = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = B.$$

Thus A and B are similar, despite them listing their eigenvalues in separate orders.

6. Prove the following: If A is invertible and similar to B, then B is invertible and A^{-1} is similar to B^{-1}.

   Solution: With A invertible, we know that its eigenvalues are nonzero, and since A is similar to B, it follows that they share the same eigenvalues. Thus B has only nonzero eigenvalues and so B is invertible.

   With A similar to B we have an invertible matrix P such that

$$A = PBP^{-1}$$

   and so if we take the inverse of both sides we find

$$A^{-1} = (PBP^{-1}) = (P^{-1})^{-1}(PB)^{-1} = P(B^{-1}P^{-1}) = PB^{-1}P^{-1}$$

   and so A^{-1} and B^{-1} are similar.

7. Define $T : \mathbf{R}^2 \to \mathbf{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}.$$

   Find a basis $\mathcal{B}$ for $\mathbf{R}^2$ with the property that $[T]_{\mathcal{B}}$ is diagonal.

   Solution: The characteristic polynomial is

$$c(\lambda) = \begin{vmatrix} -\lambda & 1 \\ -3 & 4-\lambda \end{vmatrix} = (-\lambda)(4-\lambda) - 1 \cdot (-3) = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3).$$

   Thus the eigenvalues are 1 and 3.

   For $\lambda = 1$,

$$A - I = \begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix} \quad \text{implies} \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   generates a basis for E_1.

   For $\lambda = 3$,

$$A - 3I = \begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix} \quad \text{implies} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

   generates a basis for E_3.

   Therefore a diagonalizing basis is

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\},$$

   and

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$