Discussion #18 3/9/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Problems

1. For each of the following matrices, describe in geometric terms the eigenspaces (if any) and their associated eigenvalues. *Do not compute the matrices.*

   (a) The matrix induced by the linear transformation $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ which reflects each vector across the z-axis.

   Solution: Vectors on the z-axis do not change, so they correspond to eigenspace of 1.

   Vectors in the xy plane have both of their components multiplied by -1, so these live in the eigenspace of -1. (Here we are assuming $\mathbf{e}_1 \to -\mathbf{e}_1$ and $\mathbf{e}_2 \to -\mathbf{e}_2$.)

   (b) The matrix induced by the linear transformation $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ which rotates each vector by $\pi/4$ radians counterclockwise.

   Solution: All of the vectors preserve their length under this transformation. That means $|\lambda| = 1$ for all eigenvalues. Now no vector has its orientation preserved under T and so $\lambda = 1$ is not an eigenvalue. Likewise, no vector has its direction reversed and so $\lambda = -1$ is not an eigenvalue.

   Thus this transformation has no $\mathbf{R}$-valued eigenvectors.

2. Let A be an $n \times n$ matrix. Show that A is invertible if and only if 0 is not an eigenvalue of A.

   Solution: The matrix A is invertible if and only if there is only the trivial solution to

   $$A\mathbf{x} = \mathbf{0}.$$

   This is equivalent to saying that all nonzero vectors of A get mapped to other nonzero vectors. Thus A cannot have 0 as an eigenvalue.

3. Answer the following *True* or *False*. Explain your reasoning, or give a counterexample.

   (a) Any $n \times n$ matrix that has fewer than n real distinct eigenvalues is not diagonalizable.

   Solution: False: Consider

   $$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

   then A has only one eigenvalue, 2, and is a diagonal matrix.

   (b) Eigenvectors corresponding to the same eigenvalue are always linearly dependent.

   Solution: False: Consider

   $$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

then $\mathbf{e}_1$ and $\mathbf{e}_2$ are linearly independent eigenvectors corresponding to the eigenvalue 2.

(c) If A is diagonalizable, then it has at least one eigenvalue.

Solution: True: If A is diagonalizable then it has an eigenvector, and associated with that eigenvector is an eigenvalue.

4. Show that if $b \neq 0$ then

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

is not diagonalizable.

Solution: The matrix is upper triangular and so its eigenvalues appear on the main diagonal, $\lambda = a$. If $b \neq 0$ then

$$A - aI = \begin{bmatrix} a-a & b \\ 0 & a-a \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$$

and our matrix $A - aI$ has rank 1 and nullity 1. This means there is only a one dimensional null space for $A - aI$, and thus A is not diagonalizable. For this particular A to be diagonalizable, we would require $A - aI$ to have nullity of 2.

5. (a) Find the eigenvalues, and bases for the associated eigenspaces, of

$$A = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution: Here

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 6 & 2 \\ 0 & -1-\lambda & -8 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (5-\lambda)(-1-\lambda)(2-\lambda)$$

since our matrix is upper triangular. The eigenvalues are -1, 2, and 5.

Case $\lambda = -1$: We have

$$A + I = \begin{bmatrix} 6 & 6 & 2 \\ 0 & 0 & -8 \\ 0 & 0 & 3 \end{bmatrix}$$

and so

$$\left[\begin{array}{ccc|c} 6 & 6 & 2 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1/3 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

tells us either

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

forms a basis for the eigenspace corresponding to the eigenvalue -1.

Case $\lambda = 2$: We have

$$A - 2I = \begin{bmatrix} 3 & 6 & 2 \\ 0 & -3 & -8 \\ 0 & 0 & 0 \end{bmatrix}$$

and so

$$\left[\begin{array}{ccc|c} 3 & 6 & 2 & 0 \\ 0 & -3 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2/3 & 0 \\ 0 & -3 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -14/3 & 0 \\ 0 & 1 & 8/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

tells us either

$$\mathbf{v}_2 = \begin{bmatrix} 14/3 \\ -8/3 \\ 1 \end{bmatrix} \quad \text{or} \quad \mathbf{v}_2 = \begin{bmatrix} 14 \\ -8 \\ 3 \end{bmatrix}$$

forms a basis for the eigenspace corresponding to the eigenvalue 2.

Case $\lambda = 5$: We have

$$A - 5I = \begin{bmatrix} 0 & 6 & 2 \\ 0 & -6 & -8 \\ 0 & 0 & -3 \end{bmatrix}$$

and so

$$\left[\begin{array}{ccc|c} 0 & 6 & 2 & 0 \\ 0 & -6 & -8 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & 1/3 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

tells us either

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

forms a basis for the eigenspace corresponding to the eigenvalue 5.

(b) Diagonalize the matrix from part (a).

Solution: Using the answer from part (a) we get that

$$P = \begin{bmatrix} 1 & -1 & 14 \\ 0 & 1 & -8 \\ 0 & 0 & 3 \end{bmatrix} \tag{1}$$

and

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{2}$$

6. Give an example of a 2×2 matrix with two linearly independent eigenvectors, but only one eigenvalue.

   Solution: Consider
   $$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$
   with its eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$.

7. Find a 3×3 matrix with eigenvalues $0, 1, -1$ and corresponding eigenvectors $(0, 1, 1)$, $(1, -1, 1)$, and $(0, 1, -1)$.

   Solution: Let P's columns be the desired eigenvectors and D be the diagonal matrix with the desired eigenvalues of A (appearing in the same order as the desired eigenvectors), then

$$\begin{aligned} A = PDP^{-1} &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 1/2 & -1/2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 1/2 & -1/2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1/2 & 1/2 \\ 2 & 1/2 & -1/2 \end{bmatrix}. \end{aligned}$$

8. Is
   $$\begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$$
   diagonalizable?

Solution: The matrix is lower triangular and so its eigenvalues are the diagonal entries 5, 5, and 5.

Notice that

$$A - 5I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and so $A - 5I$ has rank 2 and nullity 1. Therefore there is only one linearly independent eigenvector for $\lambda = 5$, $\mathbf{e}_3$. The matrix A is not diagonalizable.

9. (a) Suppose that $P^{-1}AP = D$, where D is diagonal. Show that $A^k = PD^kP^{-1}$ for any positive integer k.

   Solution: We have

   $$A = PDP^{-1}$$

   and so

   $$A^2 = \left(PDP^{-1}\right)\left(PDP^{-1}\right) = PDIDP^{-1} = PD^2P^{-1}$$

   while

   $$A^3 = AA^2 = A\left(PD^2P^{-1}\right) = \left(PDP^{-1}\right)\left(PD^2P^{-1}\right) = PD^3P^{-1}.$$

   The general result follows in a similar manner.

   (b) Use part (a) to compute A^{10}, where

   $$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}.$$

   Solution: We have a lower triangular matrix and so its eigenvalues are the diagonal entries 1 and 2.

   Case $\lambda = 2$: Notice

   $$A - 2I = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}$$

   tells us

   $$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

   is an eigenvector with associated eigenvalue 2.

   Case $\lambda = 1$: Notice

   $$A - I = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

   tells us

   $$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   is an eigenvector with associated eigenvalue 1.

Let

$$P = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad P^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$$

we have by part (a)

$$\begin{aligned} A^{10} &= PD^{10}P^{-1} \\ &= \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{10} \cdot \begin{bmatrix} -1/2 & 1/2 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2^{10} & 0 \\ 0 & 1^{10} \end{bmatrix} \cdot \begin{bmatrix} -1/2 & 1/2 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1{,}024 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1/2 & 1/2 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 2{,}048 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1/2 & 1/2 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -1{,}023 & 1{,}024 \end{bmatrix}. \end{aligned}$$

10. Two $n \times n$ matrices A and B are said to be *similar* if there is an invertible $n \times n$ matrix P such that $B = P^{-1}AP$. Show that similar matrices always have the same eigenvalues. Must they have the same eigenvectors?

Solution: Let $B = P^{-1}AS$ then $P^{-1}P = I$ tells us

$$\begin{aligned} B - \lambda I &= P^{-1}AP - \lambda I \cdot I \\ &= P^{-1}AP - \lambda(P^{-1}P)I \\ &= P^{-1}AP - \lambda P^{-1}IP \\ &= P^{-1}(AP - \lambda IP) \\ &= P^{-1}(A - \lambda I)P \end{aligned}$$

so by our product rule for determinants

$$\begin{aligned} \det(B - \lambda I) &= \det(S^{-1}(A - \lambda I)S) \\ &= \det\left(S^{-1}\right) \cdot \det(A - \lambda I) \cdot \det(S) \\ &= \underbrace{\det\left(S^{-1}\right) \cdot \det(S)}_{\text{equals } 1} \cdot \det(A - \lambda I) \\ &= 1 \cdot \det(A - \lambda I) \\ &= \det(A - \lambda I) \end{aligned}$$

implies A and B have the same characteristic polynomial. Thus A and B have the same eigenvalues.

They need not have the same eigenvectors, notice that the matrix from problem 2

$$A = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 0 & 0 & 2 \end{bmatrix}$$

is similar to

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

where S is a matrix whose column vectors are the eigenvectors of A. Then

$$\mathbf{e}_2$$

is an eigenvector of D, but is not an eigenvector of A.