Discussion #17 3/6/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Eigenvectors and eigenvalues play an important role in computational mathematics. Eigenvectors have a simple description, we need a nonzero vector $\mathbf{v} \in \mathbf{R}^n$ such that for $A \in \mathbf{R}^{n\times n}$ we have

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some $\lambda \in \mathbf{R}$. Eigenvectors are not just limited to vector spaces such as $\mathbf{R}^n$, the differentiation operator has its own eigenfunctions and they play a fundamental role in a later coursework.

But before we get into that, let's do a little change of basis stuff.

Questions

1. Let $\mathbb{R}^2$ have the standard basis $\mathcal{E} = \{\mathbf{e_1}, \mathbf{e_2}\}$ and a non-standard basis $\mathcal{B} = \{\mathbf{b_1}, \mathbf{b_2}\}$, where

$$\mathbf{b_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{b_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

   (a) Find the change of basis matrix $P_{\mathcal{E}\leftarrow\mathcal{B}}$.

   (b) Find the change of basis matrix $P_{\mathcal{B}\leftarrow\mathcal{E}}$.

   (c) If a vector $\mathbf{x}$ has coordinates $[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, find $[\mathbf{x}]_E$.

2. Consider two bases for $\mathbb{R}^2$, $\mathcal{B} = \{\mathbf{u_1}, \mathbf{u_2}\}$ and $\mathcal{C} = \{\mathbf{v_1}, \mathbf{v_2}\}$, defined by:

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   (a) Find the change of basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$.

   (b) Find the change of basis matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$.

   (c) If $\mathbf{w} = \mathbf{u_1} + 2\mathbf{u_2}$, what is $[\mathbf{w}]_{\mathcal{C}}$?

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation whose matrix relative to the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$ is:

$$[T]_{\mathcal{E}} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

   Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^2$.

   (a) Find the change of basis matrix P from $\mathcal{B}$ to $\mathcal{E}$.

   (b) Use the similarity transformation formula $[T]_{\mathcal{B}} = P^{-1}[T]_{\mathcal{E}}P$ to find the matrix of T relative to the basis $\mathcal{B}$.

4. Answer the following *True* or *False*. Explain your reasoning, or give a counterexample.

   (a) The sum of two eigenvalues of a matrix A is also an eigenvalue of A.

   Solution: As outlined in part (a), the eigenvalues of I must be 1, and so $2 = 1 + 1$ is not an eigenvalue of I.

   (b) The sum of two eigenvectors of a matrix A is also an eigenvector of A.

   Solution: False: If $\mathbf{v}$ and $\mathbf{w}$ are non zero eigenvectors with

$$A\mathbf{v} = 3\mathbf{v} \quad \text{and} \quad A\mathbf{w} = 2\mathbf{w}$$

   then

$$A(\mathbf{v} + \mathbf{w}) = 3\mathbf{v} + 2\mathbf{w} \neq \lambda(\mathbf{v} + \mathbf{w})$$

   for any $\lambda \in \mathbf{R}$.

   This will hold if the eigenvectors are associated with the same eigenvalues. Or a collection of all possible eigenvectors associated with the eigenvalue λ for a vector subspace.

   (c) There exists a square matrix with no real eigenvalues.

   Solution: True: Consider

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

   then

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = (-\lambda)^2 - (-1) \cdot 1 = \lambda^2 + 1.$$

   The eigenvalues are $\lambda = \pm i$.

   (d) There exists an $n \times n$ matrix with $n + 1$ distinct eigenvalues.

   Solution: False: The characteristic polynomial

$$\det(A - \lambda I)$$

   is of order n and so it has at most n distinct roots. This tells us A has at most n distinct eigenvalues.

5. Suppose D is the differentiation operator on the space of differentiable functions over $\mathbf{R}$,

$$Df = \frac{d}{dx} f(x) = f'(x).$$

   Can you find a function f such that

$$f \neq 0 \quad \text{and} \quad Df = \lambda f$$

   for some $\lambda \in \mathbf{R}$? Is your function f unique?

Solution: We have

$$\frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x}$$

for all $\lambda \in \mathbf{R}$. Thus we have infinitely many eigenfunctions for the operator D.

6. Given $\lambda = 6$ and $\lambda = -1$ are eigenvalues of

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}$$

find a basis for each eigenspace of A.

Solution: Case $\lambda = -1$:

We have

$$A - (-1)I = A + I = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

tells us

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

is an eigenvector corresponding to eigenvalue -1.

Case $\lambda = 6$:

We have

$$A - 6I = \begin{bmatrix} -4 & 3 \\ 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/4 \\ 0 & 0 \end{bmatrix}$$

tells us

$$\mathbf{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

is an eigenvector corresponding to eigenvalue 6. (This vector was chosen to avoid fractions.)

Thus

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

each span their corresponding eigenspaces.

7. Is 5 an eigenvalue of

$$A = \begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix}?$$

Solution: If 5 is an eigenvalue of A then there exists nontrivial solutions to

$$(A - 5I)\mathbf{x} = \mathbf{0}.$$

Notice

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 3 & -5 & 5 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 8 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 5/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

and since there is a pivot in every column we only have the trivial solution. Thus 5 is **not** an eigenvalue of A.

8. Find the characteristic polynomial and the eigenvalues of the matrix

$$\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}.$$

Solution: Our characteristic polynomial is

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{vmatrix} \\ &= (2-\lambda)(2-\lambda) - 7 \cdot 7) \\ &= \lambda^2 - 4\lambda + 4 - 49 \\ &= \lambda^2 - 4\lambda - 45 \\ &= (\lambda - 9)(\lambda + 5) \end{aligned}$$

and this tells us our eigenvalues are $\lambda = -5$ and $\lambda = 9$.

9. List the eigenvalues, repeated according to their multiplicities, of the matrix

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 1 & -5 & 2 & 1 \end{bmatrix}.$$

Solution: The determinant of a lower/upper triangular matrix is the product of the diagonal entries and so

$$A - \lambda I = \begin{vmatrix} 5-\lambda & 0 & 0 & 0 \\ 8 & -4-\lambda & 0 & 0 \\ 0 & 7 & 1-\lambda & 0 \\ 1 & -5 & 2 & 1-\lambda \end{vmatrix} = \underbrace{(5-\lambda)(-4-\lambda)(1-\lambda)(1-\lambda)}_{\text{do not expand further}}.$$

Thus our eigenvalues of A are -4, 1, 1, and 5.

10. Show that if A^2 is the zero matrix, then the only eigenvalue of A is 0. Must A be the zero matrix?

Solution: Suppose $\mathbf{x}$ is an nonzero eigenvector of A,

$$A\mathbf{x} = \lambda\mathbf{x}$$

where $\lambda \in \mathbf{R}$ then

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}.$$

While

$$A^2\mathbf{x} = \mathbf{0} \text{ implies } \mathbf{0} = \lambda^2\mathbf{x}$$

tells us $\lambda^2 = 0$.

Consider

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

then

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence A need not be the zero matrix.