Discussion #15 3/2/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

The **rank** of a matrix is the dimension of the column space. It is the number of pivot columns of the matrix (after row reducing). The **nullity** of a matrix is the dimension of the null space. It is the number of *non*-pivot column of the matrix (after row reducing).

To calculate the dimension of a subspace, find a spanning set for the subspace, put it into a matrix, row reduce, and count the number of pivot columns.

Problems

1. Answer the following *True* or *False*. Justify your answers.

   (a) $\dim(\mathbf{P}_5) = 5$.

   Solution: False:
   $$\dim(\mathbf{P}_5) = 6$$
   because the standard basis for $\mathbf{P}_5$
   $$\{1, x, x^2, x^3, x^4, x^5\}$$
   has 6 elements.

   (b) $C[0, 1]$ is infinite dimensional. (Hint: See Theorem 10, page 241 of Lay.) If this part is confusing, skip it until later.

   Solution: True: The subspace of polynomials, of any degree, has infinitely many terms in its basis. Thus $C[0, 1]$ is infinite dimensional.

   (c) If W is a subspace of V, and S is a basis for W, then we can add vectors to S to form a basis for V.

   Solution: True: If V is finite dimensional, we can eventually add more and more elements from V's basis to span all of V. In the infinite dimensional case, it can be done too, but requires the axiom of choice.

   (d) If W is a subspace of V, and S is a basis for V, then some subset of S is a basis for W.

   Solution: False: Let
   $$W = \text{Span}\{(1, 1)\}$$
   and
   $$S = \{\mathbf{e}_1, \mathbf{e}_2\}$$
   then no subset of S spans W, a line in $\mathbf{R}^2$, through the origin with slope 1.

2. (a) Give an example of a 3×3 matrix whose null space has dimension 1.

Solution:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1}$$

(b) Give an example of a 3×3 matrix whose column space has dimension 1.

Solution:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2}$$

(c) Does there exist a 3×3 matrix whose null space and column space both have dimension 1?

Solution: No. By rank-nullity the dimensions have to add up to the number of columns. But $1 + 1 \neq 3$.

3. Let A be an $m \times n$ matrix.

(a) If A is onto, what is the rank of A?

Solution: A has a pivot in every row, so the rank is m. Alternatively, the image of A is all of $\mathbb{R}^m$, which is m-dimensional, so the rank is m.

(b) If A is one-to-one, what is the nullity of A?

Solution: A has a pivot in every column, so there are no non-pivot columns, so the nullity is 0. Alternatively, $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, so the null space is just the zero vector, which is 0-dimensional.

4. Suppose A is an invertible $n \times n$ matrix.

(a) What is $\text{rank}(A)$?

Solution: If A is invertible it has full rank, $\text{rank}(A) = n$.

(b) What is $\dim(\text{Nul}(A))$?

Solution: We know that

$$\text{rank}(A) + \text{nullity}(A) = n$$

and

$$\text{nullity}(A) = \dim(\text{Nul}(A))$$

and so with A having full rank, $\text{nullity}(A) = 0$.

5. Show that if A is not square, then either the rows of A or the columns of A are linearly dependent.

Solution: If $A \in \mathbf{R}^{m\times n}$ and $n > m$, then A has more columns than rows. Since the columns of A lie in $\mathbf{R}^m$, and you cannot have more than m linearly independent vectors in $\mathbf{R}^m$, the columns of A must be linearly dependent.

If $m > n$, then $A^T \in \mathbf{R}^{n\times m}$ has more columns than rows. The columns of A^T, which are the rows of A, must be linearly dependent.

6. Let $W = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\}$.

   (a) Find a matrix A such that $\mathrm{Nul}(A) = W$.

   Solution: If we choose a matrix in reduced row echelon form that honors the condition
   $$x_1 + x_2 + x_3 = 0$$
   we are done. So
   $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
   would give us the system matrix
   $$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

   (b) Use part (a) to find a basis for W and determine its dimension.

   Solution: We have
   $$\mathbf{x} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$
   and so we can take
   $$\{(-1, 1, 0), (-1, 0, 1)\}$$
   as our basis for W. The dimension of W is 2.

7. Let V be the vector space consisting of all polynomials $p(x)$ of degree 3 or less, satisfying $p(1) = 0$. What is $\dim(V)$? Give a basis for V.

   Solution: We want polynomials of the form
   $$a(x-1)^3 + b(x-1)^2 + c(x-1)$$
   and so we can choose
   $$\{x - 1, (x-1)^2, (x-1)^3\}$$
   as our basis for V. The dimension of V is 3.

8. (a) Find a basis for the vector space of all 3×3 matrices. What's the dimension?

Solution: Consider

$$E_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_{31} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_{33} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the dimension is 9.

(b) Find a basis for the vector space of all 3×3 symmetric matrices. What is the dimension of this vector space?

Solution: Consider

$$S_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad S_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad S_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$S_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad S_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$S_{33} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the dimension is 6.