Discussion #5 2/2/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Problems

1. Find a basis for the subspace W of $\mathbf{R}^4$ spanned by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 6 \\ 15 \end{bmatrix}$$

Solution: Consider

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 3 \\ 0 & 2 & 6 \\ 4 & 1 & 15 \end{bmatrix}$$

and

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -2 & 3 & 3 & 0 \\ 0 & 2 & 6 & 0 \\ 4 & 1 & 15 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 5 & 15 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

tells us that the vectors are linearly dependent. We only need the first two vectors to form a basis because they correspond to pivot columns in A's reduced row echelon form. Our desired basis for W is

$$\{\mathbf{v}_1, \mathbf{v}_2\}.$$

2. Show that $\{x^3 + x + 1, 2x^3 + x + 1, x^3 + 3x + 1, x^3 + x + 4\}$ is a linearly dependent set in $\mathbf{P}_3$. (Hint: Find a 3-dimensional subspace that they all lie in.)

Solution: We can identify these vectors in $\mathbf{R}^3$, by ignoring the missing second order terms, and realize that we have 4 vectors in a 3 dimension space. This means they must be linearly dependent.

Or we can embed them in $\mathbf{R}^4$, just keeping an extra row of zeros in our system matrix, and row reduce (or just recognize that we cannot have a pivot in every column):

$$x^3+x+1 \equiv \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \quad 2x^3+x+1 \equiv \begin{bmatrix}1\\1\\0\\2\end{bmatrix},$$

$$x^3+3x+1 \equiv \begin{bmatrix}1\\3\\0\\1\end{bmatrix}, \quad x^3+x+4 \equiv \begin{bmatrix}4\\1\\0\\1\end{bmatrix}$$

and so

$$\left[\begin{array}{cccc|c}1&1&1&4&0\\1&1&3&1&0\\0&0&0&0&0\\1&2&1&1&0\end{array}\right] \sim \left[\begin{array}{cccc|c}1&1&1&4&0\\0&0&2&-3&0\\0&0&0&0&0\\0&1&0&-3&0\end{array}\right]$$
$$\sim \left[\begin{array}{cccc|c}1&1&1&4&0\\0&1&0&-3&0\\0&0&0&0&0\\0&0&2&-3&0\end{array}\right]$$
$$\sim \left[\begin{array}{cccc|c}1&0&1&7&0\\0&1&0&-3&0\\0&0&0&0&0\\0&0&2&-3&0\end{array}\right]$$
$$\sim \left[\begin{array}{cccc|c}1&0&1&7&0\\0&1&0&-3&0\\0&0&2&-3&0\\0&0&0&0&0\end{array}\right]$$
$$\sim \left[\begin{array}{cccc|c}1&0&0&17/2&0\\0&1&0&-3&0\\0&0&1&-3/2&0\\0&0&0&0&0\end{array}\right]$$

tells us we have lack a pivot in every column. Thus the 4 polynomials are linearly dependent in $\mathbf{P}_3$.

3. (a) Show that any two vectors chosen from a linearly independent set are linearly independent.

   Solution: Let $c_1, c_2 \in \mathbf{R}$ then with the entire set linearly independent

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \sum_{k=3}^{m} 0\mathbf{v}_k = \mathbf{0}$$

if and only if $c_1 = c_2 = 0$. Thus $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

(b) Show that a set which contains two linearly dependent vectors must be a linearly dependent set.

Solution: We have

$$\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$$

where $c_2 \neq 0$ and so

$$\mathbf{v}_1 + c_2\mathbf{v}_2 + \sum_{k=3}^{m} 0\mathbf{v}_k = \mathbf{0}$$

gives a nontrival solution to the system of equations. Thus the entire set is linearly dependent.

(c) Find three vectors in $\mathbf{R}^3$ which are linearly dependent, and such that any two of them are linearly independent.

Solution: Choose

$$\mathbf{e}_1, \quad \mathbf{e}_2, \quad \text{and} \quad (\mathbf{e}_1 + \mathbf{e}_2).$$

4. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \end{bmatrix} \right\}, \quad [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}. \tag{1}$$

What is the vector $\mathbf{x}$ given by the coordinates $[\mathbf{x}]_{\mathcal{B}}$?

5. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix} \right\}, \quad [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 8 \\ -7 \end{bmatrix}. \tag{2}$$

What is the vector $\mathbf{x}$ given by the coordinates $[\mathbf{x}]_{\mathcal{B}}$?

6. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \end{bmatrix} \right\}, \quad \mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{3}$$

What are the coordinates of $\mathbf{x}$ with respect to the basis $\mathcal{B}$?

7. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix} \right\}, \mathbf{x} = \begin{bmatrix} 4 \\ -7 \\ 4 \end{bmatrix}. \tag{4}$$

What are the coordinates of $\mathbf{x}$ with respect to the basis $\mathcal{B}$?