Discussion #12 2/23/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Problems

1. Consider the matrix
$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & -1 & -2 \end{bmatrix}.$$
   (a) Find a spanning set for the null space of A.

   (b) Find a spanning set for the column space of A. Can you find a spanning set with only 2 vectors?

2. Let A be an $n \times n$ matrix such that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

   (a) What is $\operatorname{Nul} A$?

   (b) What is $\operatorname{Col} A$?

3. Consider the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ with standard matrix representation
$$A = \begin{bmatrix} 2 & 1 & -1 \\ -4 & -2 & 2 \end{bmatrix}.$$
   (a) Carefully sketch the null space of A. (Of what vector space is it a subspace?)

   Solution: We have
$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ -4 & -2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
   and so
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/2x_2 + 1/2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}.$$
   Thus
$$\operatorname{Nul}(A) = \operatorname{Span}\{(-1/2, 1, 0), (1/2, 0, 1)\}$$
   is a plane in $\mathbf{R}^3$.

   (b) Carefully sketch the column space of A. (Of what vector space is it a subspace?)

   Solution: The column space is the span of the pivot columns in A, so the first column,
$$\operatorname{Col}(A) = \operatorname{Span}\{(2, -4)\}$$
   and this forms a line in $\mathbf{R}^2$.

4. Let
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) Show that the null space of A is the z-axis and the column space of A is the xy-plane.

Solution: We have

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and so

$$\text{Nul}(A) = \text{Span}\{(0,0,1)\}$$

which corresponds to the z-axis. While the column space of A is the span of A's pivot columns, the first two columns,

$$\text{Col}(A) = \text{Span}\{(0,1,0),(1,0,0)\}.$$

and this is the xy-plane.

(b) Find a 3×3 matrix whose null space is the x-axis and whose column space is the yz-plane.

Solution: Consider

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

then

$$A\mathbf{e}_1 = \mathbf{0} \quad \text{and} \quad A(c\mathbf{e}_2 + d\mathbf{e}_3) = c\mathbf{e}_2 + d\mathbf{e}_3.$$

(c) Find a matrix whose row space is spanned by $(1,0,1)$ and $(0,1,0)$ and whose null space is the span of $(1,0,-1)$.

Solution: Consider

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

then

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and notice we have two pivots.

Now

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

gives us the desired null space, while A's row space is the span of A's pivot rows, the first two rows,

$$\text{Col}(A) = \text{Span}\{(1,0,1),(0,1,0)\}.$$

5. (a) Give an example of a 3×3 matrix whose null space has dimension 1.

   (b) Give an example of a 3×3 matrix whose column space has dimension 1.

   (c) Does there exist a 3×3 matrix whose null space and column space both have dimension 1?

6. (Lay 4.1.40) Let H and K be subspaces of a vector space V. The intersection of H and K, written as $H \cap K$, is the set of $v \in V$ that belong to both H and K.Show that $H \cap K$ is a subspace of V. Give an example in $\mathbb{R}^2$ to show that the union of two subspaces is not, in general, a subspace.

7. Let A be a $n \times n$ matrix such that $A^2 = 0$. Show that $\operatorname{Col} A$ is a subspace of $\operatorname{Nul} A$.