Discussion #10 2/13/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Problems

1. Determine if V is closed under addition, scalar multiplication, both, or neither.

   (a) $V = \{\text{odd integers}\}$

   Solution: Take $1, 3 \in V$ then $4 \notin V$ implies the set is not closed under addition. Let $x = 1$ and $c = 1/2$ then cx is not odd and so V is not closed under scalar multiplication.

   (b) $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \text{ are negative} \right\}$

   Solution: If $a < 0$ and $b < 0$ then $a + b < 0$. This shows that the space V is closed under addition.

   Notice that
   $$A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$
   is in V but $-A \notin V$. Thus the space is not closed under scalar multiplication.

   (c) $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is nonsingular} \right\}$

   Solution: The matrices
   $$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
   and $-I$ are invertible but $(I - I) \notin V$ since the zero matrix is not invertible. Likewise, $0I$ is not in V so the space is not closed under addition or scalar multiplication.

   (d) $V = \{f \in C[1, 2] : f(x) \geq 0 \text{ for all } x \in [1, 2]\}$

   Solution: If $f, g \in C[1, 2]$ then $f + g$ is continuous on $[1, 2]$ and
   $$f(x) \geq 0 \quad \text{and} \quad g(x) \geq 0 \quad \text{implies} \quad f(x) + g(x) \geq 0$$
   so the space is closed under addition. However
   $$f(x) = 1$$
   for all $x \in [1, 2]$ is in V but $-f \notin V$. This shows the space is not closed under scalar multiplication.

2. What special properties does the zero vector of a vector space have? Are there other vectors with these same properties? You should try to prove your answer.

Solution: The zero vector is unique. Suppose

$$\mathbf{v} + \mathbf{w} = \mathbf{v}$$

then we have

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$$

and substituting $\mathbf{v} = \mathbf{v} + \mathbf{w}$ gives

$$\begin{aligned}(\mathbf{v} + \mathbf{w}) + (-\mathbf{v}) &= \mathbf{0}\\ \mathbf{v} + (-\mathbf{v}) + \mathbf{w} &= \mathbf{0}\\ \mathbf{0} + \mathbf{w} &= \mathbf{0}\\ \mathbf{w} &= \mathbf{0}.\end{aligned}$$

Hence any vector that acts like the zero vector must be the zero vector.

3. Let V be the set of all solutions to the differential equation $y'' - 4y' = 0$.

   (a) Is 0 in V?

   Solution: Yes, if $y = 0$ for all x then

   $$y'' - 4y' = 0 - 4 \cdot 0 = 0.$$

   (b) If y is a solution to the differential equation, is $2y$ also a solution?

   Solution: Yes, if y satisfies

   $$y'' - 4y' = 0$$

   then

   $$2y'' - 8y' = 0 \quad \text{if and only if} \quad (2y)'' - 4(2y)' = 0.$$

   (c) If y_1 and y_2 are solutions, is $2y_1 + 3y_2$ also a solution?

   Solution: Yes, if y_1 and y_2 are solutions, so are $2y_1$ and $3y_2$ by an argument similar to (b). Then

   $$(2y_1)'' - 4(2y_1)' = 0 \quad \text{and} \quad (3y_2)'' - 4(3y_2)' = 0$$

   and add the two equations together

   $$\begin{aligned}(2y_1)'' + (3y_2)'' - 4(2y_1)' - 4(3y_2)' &= 0\\ (2y_1 + 3y_2)'' - 4(2y_1 + 3y_2)' &= 0\end{aligned}$$

   tells us $2y_1 + 3y_2$ is a solution.

   (d) Verify that V is a vector space.

Solution: All we have to do is repeat the arguments from (b) and (c) but in a more general setting. Suppose y_1 and y_2 are solutions and $c_1, c_2 \in \mathbf{R}$.

Then

$$(c_1y_1 + c_2y_2)' = c_1y_1' + c_2y_2' \quad \text{and} \quad (c_1y_1 + c_2y_2)'' = c_1y_1'' + c_2y_2''$$

and so

$$(c_1y_1 + c_2y_2)'' - 4(c_1y_1 + c_2y_2)' = c_1y_1'' - 4c_1y_1' + c_2y_2'' - 4c_2y_2' = 0 + 0 = 0$$

tells us $c_1y_1 + c_2y_2$ is a solution.

With the space closed under addition and scalar multiplication. the rest of the axioms for a vector space are satisfied. The majority of the work in showing this comes from the fact that addition is well behaved.

4. (a) Let V be a vector space. Is V a subspace of itself?

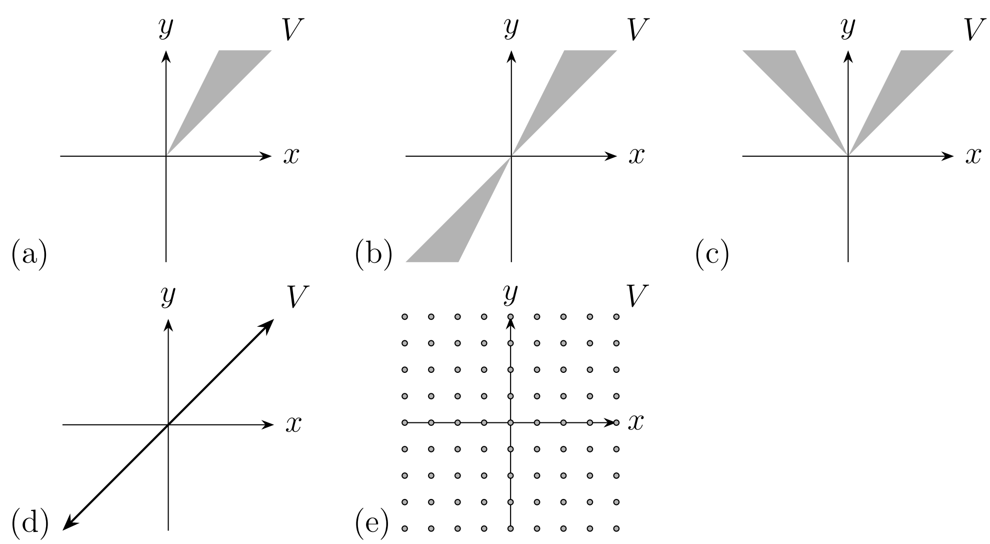
   Solution: Yes, it is a nonempty subset of itself and is clearly closed under addition and scalar multiplication. If you want to be precise, it is subspace of V, but not a proper subspace of V since, $V = V$. Proper subspaces are not equal to the entire vector space V.

   (b) If $\mathbf{0}$ is the zero vector in V, is the set $\{\mathbf{0}\}$ a subspace of V?

   Solution: Yes, it is the trivial subspace of V. Is it closed under scalar multiplication and vector addition since

$$\mathbf{0} + \mathbf{0} = \mathbf{0} \quad \text{and} \quad 0 \cdot \mathbf{0} = \mathbf{0}.$$

5. Which regions V are subspaces of $\mathbf{R}^2$?



Solution:

(a) **Not** a subspace: Take $\mathbf{v} \in V$ then $-\mathbf{v} \notin V$.

(b) **Not** a subspace: This space is closed under scalar multiplication if we assume the set V is infinitely long. For the sake the argument, assume the two boundaries for the grey region are lines given by

$$y = x \quad \text{and} \quad y = 2x.$$

Thus

$$(1, 1), (2, 1) \in V$$

but

$$(1, 1) - (2, 1) = (-1, 0) \notin V$$

so the space is not closed under vector addition.

(c) **Not** a subspace: take $\mathbf{v} \in V$ then $-\mathbf{v} \notin V$.

(d) **Is** a subspace: We have $V = \text{Span}\{\mathbf{e}_1\}$.

(e) **Not** a subspace: Assume these dots appear at (n, n) where $n \in \mathbf{Z}$. Then

$$1/2 \cdot (n, n) \notin V$$

so the set is not closed under scalar multiplication over $\mathbf{R}$. (If you restricted where the scalars could come from, say in $\mathbf{Z}$, you could make this a vector space. However, this is a topic explored in more advanced linear algebra courses.)

6. Let V be the vector space of all 2×2 matrices, and let

$$W = \text{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$$

(a) Is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ in W?
Solution: Yes,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) Is $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ in W?
Solution: No, any vector in W must be of the form

$$a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

where $a, b \in \mathbf{R}$.

(c) Is W a subspace of V?

Solution: Yes, a nonempty span of vectors is automatically a subspace.

(d) What does a typical vector in W look like?

Solution: Any vector in W must be of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

where $a, b \in \mathbf{R}$.

7. Suppose that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ spans the vector space V, and for each i, that $\mathbf{v}_i$ lies in $\operatorname{Span}\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$. Show that $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ spans V.

Note: This can be a very useful tool! Suppose you want to know if the space spanned by $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is the same as the space spanned by $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$. If you can show that each $\mathbf{v}_i$ is in $\operatorname{Span}\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ and each $\mathbf{w}_i$ is in $\operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then the spans must be the same. Make sure you understand this concept before moving on.

Solution: Let $\mathbf{b} \in V$ then we have $c_1, \ldots, c_n \in \mathbf{R}$ such that

$$\mathbf{b} = \sum_{j=1}^{n} c_j \mathbf{v}_j.$$

because $\operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = V$. For each $\mathbf{v}_j$ we have $a_{j1}, a_{j2}, \ldots, a_{jm} \in \mathbf{R}$ such that

$$\mathbf{v}_j = \sum_{k=1}^{m} a_{jk} \mathbf{w}_k.$$

Hence

$$\mathbf{b} = \sum_{j=1}^{n} c_j \left(\sum_{k=1}^{m} a_{jk} \mathbf{w}_k \right) = \sum_{k=1}^{m} \sum_{j=1}^{n} c_j a_{jk} \mathbf{w}_k$$

because we can swap the order of two finite sums. This expresses $\mathbf{b}$ as a linear combination of $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ so

$$\mathbf{b} \in \operatorname{Span}\{\mathbf{w}_1, \ldots, \mathbf{w}_m\} \quad \text{and} \quad \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = V.$$