Discussion #8 2/9/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

The Invertible Matrix Theorem, Lay 2.3 Theorem 8 (p. 145), is very very useful. You don't need to memorize all the parts of it, but you should know what they mean.

Determinants can be confusing, but once you learn to calculate them and a few of their basic properties, you'll know all you need to know.

Problems

1. Let A be a square matrix. List at least four different statements which are equivalent to the statement
$$\det(A) \neq 0.$$

   Solution:

   1. The matrix A is invertible.
   2. There is only one solution to $A\mathbf{x} = \mathbf{0}$.
   3. The columns of A are linearly independent.
   4. The matrix A is 1-1.
   5. The matrix A is onto.
   6. For each $\mathbf{b} \in \mathbf{R}^n$, there exists a unique solution to $A\mathbf{x} = \mathbf{b}$.

   $\vdots$

2. Let A and B be $n \times n$ matrices. Answer the following *True* or *False*. If *False* give a counterexample.

   (a) $\det(AB) = \det(A) \cdot \det(B)$

   Solution: **True**: This holds as shown in §3.2 Theorem 6.

   (b) $\det(AB) = \det(BA)$

   Solution: **True**: We have
$$\det(AB) = \det(A) \cdot \det(B) = \det(B) \cdot \det(A) = \det(BA).$$

   (c) $\det(A + B) = \det(A) + \det(B)$

   Solution: **False**: Consider $I \in \mathbf{R}^{2\times 2}$:
$$\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 \quad \text{and} \quad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

and so
$$4 = \det(2I) \neq \det(I) + \det(I) = 1 + 1 = 2.$$

(d) If A and B are both invertible, then AB is invertible.

Solution: True: We have
$$\det(AB) = \det(A) \cdot \det(B) \neq 0$$
and so AB is invertible.

(e) $\det(A) = \det\left(A^T\right)$

Solution: True: This is Theorem 5 in §2.2.

(f) If A is invertible, then $\det(A^{-1}) = (\det(A))^{-1}$.

Solution: True: We have
$$\det(I) = 1 \quad \text{and} \quad \det(I) = \det(AA^{-1}) = \det(A) \cdot \det(A^{-1})$$
and so with $\det(A) \neq 0$
$$\det(A^{-1}) = (\det(A))^{-1}.$$

(g) If c is a scalar then $\det(cA) = c^n \det(A)$.

Solution: True: This is n applications of Theorem 3 part (c) in §3.2.

3. What is the volume of a parallelogram with vertices at $(0, 0)$, $(4, 1)$, $(3, 5)$, $(7, 6)$?

Solution: We take the determinant of the matrix with the vertices as the entries:
$$\det \begin{bmatrix} 4 & 3 \\ 1 & 5 \end{bmatrix} = 17. \tag{1}$$

4. Suppose A is a 3×3 matrix with determinant 5. What is $\det(3A)$? $\det(A^{-1})$? $\det(2A^{-1})$? $\det((2A)^{-1})$?

Solution: We have
$$\det(3A) = 3^3 \det(A) = 27 \cdot 5 = 135, \quad \det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{5},$$
and
$$\det(2A^{-1}) = 2^3 \det(A^{-1}) = \frac{8}{5}.$$
While
$$(2A)^{-1} = \frac{1}{2} A^{-1}$$

since

$$(2A)\frac{1}{2}A^{-1} = \frac{2}{2}AA^{-1} = I$$

so

$$\det((2A)^{-1}) = \det\left(\frac{1}{2}A^{-1}\right) = \frac{1}{2^3}\cdot\frac{1}{5} = \frac{1}{40}.$$

5. Compute the following:

(a) $\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & -1 & -2 \\ 3 & 0 & 1 \\ -1 & 1 & 1 \end{vmatrix}$ (c) $\begin{vmatrix} \sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix}$

Solution:

(a) Here

$$\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = 1\cdot 5 - 2\cdot 3 = -1.$$

(b) Here we expand about the second row

$$\begin{aligned}
\begin{vmatrix} 1 & -1 & -2 \\ 3 & 0 & 1 \\ -1 & 1 & 1 \end{vmatrix} &= (-1)^{2+1}\cdot 3\cdot\begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix} + (-1)^{2+2}\cdot 0\cdot\begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} \\
&\quad + (-1)^{2+3}\cdot 1\cdot\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \\
&= (-3)\cdot((-1)\cdot 1 - (-2)\cdot 1) + 0 - (1\cdot 1 - (-1)\cdot(-1)) \\
&= (-3)\cdot 1 - 0 \\
&= -3.
\end{aligned}$$

(c) We expand about the 3rd row

$$\begin{aligned}
\begin{vmatrix} \sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} &= (-1)^{3+1}\cdot 0\cdot\begin{vmatrix} \cos(\theta) & 0 \\ \sin(\theta) & 0 \end{vmatrix} + (-1)^{3+2}\cdot 0\cdot\begin{vmatrix} \sin(\theta) & 0 \\ -\cos(\theta) & 0 \end{vmatrix} \\
&\quad + (-1)^{3+3}\cdot 1\cdot\begin{vmatrix} \sin(\theta) & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) \end{vmatrix} \\
&= 1\cdot(\sin^2(\theta) + \cos^2(\theta)) \\
&= 1.
\end{aligned}$$

6. By inspection, evaluate the following determinants:

(a) $\begin{vmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 5 & 9 \end{vmatrix}$ (b) $\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 9 \\ 0 & 16 & 25 & 36 \\ 49 & 64 & 81 & 100 \end{vmatrix}$ (c) $\begin{vmatrix} 0 & \pi & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 22 \end{vmatrix}$

Solution:

(a) We have

$$\begin{vmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 5 & 9 \end{vmatrix} = 3 \cdot 4 \cdot 9 = 54$$

because we have a lower triangular matrix.

(b) If we swap R_1 and R_4, the R_2 and R_3 we get an upper triangular matrix

$$\begin{vmatrix} 0 & \pi & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 22 \end{vmatrix} = \begin{vmatrix} 49 & 64 & 81 & 100 \\ 0 & 16 & 25 & 36 \\ 0 & 0 & 4 & 9 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 49 \cdot 16 \cdot 4 \cdot 1$$

and the two row swaps cancel out the sign change in the determinant. Thus this also gives the determinant of our matrix.

(c) We perform 2 row swaps, R_1 and R_3, then R_2 and R_3, which do not change the value of the determinant, and then get

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \pi & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 22 \end{vmatrix} = 1 \cdot \pi \cdot (\sqrt{2}) \cdot 22.$$

7. Show that if A is a square matrix with a row of zeros, then $\det(A) = 0$. What if A has a column of zeros?

Solution: We can perform a row swap to get B with a row of zeros on the top then

$$\det(A) = -\det(B)$$

but then we expand $\det(B)$ along the top row and get

$$\det(B) = 0$$

and thus $\det(A) = 0$. If A has a column of zeros we can use

$$\det(A) = \det\left(A^T\right) = 0$$

since A^T now has a row of all zeros.

8. (a) Show that the equation $A\mathbf{x} = \mathbf{x}$ can be rewritten as

$$(A - I)\,\mathbf{x} = 0,$$

where A is an $n \times n$ matrix and I is the $n \times n$ identity matrix.

Solution: To get matrix I to appear, we will not that $\mathbf{x} = I\mathbf{x}$. Then

$$\begin{aligned} A\mathbf{x} &= \mathbf{x} \\ A\mathbf{x} &= I\mathbf{x} \\ A\mathbf{x} - I\mathbf{x} &= \mathbf{0} \\ (A - I)\,\mathbf{x} &= 0. \end{aligned}$$

(b) Use part (a) to solve $A\mathbf{x} = \mathbf{x}$ for $\mathbf{x}$, where

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & -1 \\ 2 & -2 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution: Here

$$A - I = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & -1 \\ 2 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{bmatrix}$$

and so

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 1 & -1 & 0 \\ 2 & -2 & 0 & 0 \end{array}\right] &\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & -6 & -6 & 0 \end{array}\right] \\ &\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & 18 & 0 \end{array}\right] \\ &\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \\ &\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \\ &\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \\ &\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \end{aligned}$$

tells us we only have the trivial solution

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(c) Solve $A\mathbf{x} = 4\mathbf{x}$.

Solution: Inspired by (a), we want to solve

$$(A - 4I)\mathbf{x} = \mathbf{0}$$

where

$$A - 4I = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & -1 \\ 2 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -2 & -1 \\ 2 & -2 & -3 \end{bmatrix}.$$

Now

$$\left[\begin{array}{ccc|c} -2 & 2 & 3 & 0 \\ 1 & -2 & -1 & 0 \\ 2 & -2 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -2 & 2 & 3 & 0 \\ 0 & -1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} -2 & 2 & 3 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} -2 & 0 & 4 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

tells us that our solution is of the form

$$\mathbf{x} = x_3 \begin{bmatrix} 2 \\ 1/2 \\ 1 \end{bmatrix}$$

where $x_3 \in \mathbf{R}$.