Discussion #7 2/6/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Problems

1. Answer the following with *True* or *False*. Explain your reasoning, or give a counterexample.

   (a) If A and B are any matrices, then $A + B$ is defined.

   Solution: False: Consider

   $$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

   (b) If A and B are both $n \times n$ matrices, then $A + B = B + A$.

   Solution: True: The order of scalar addition does not matter,

   $$(A + B)_{ij} = A_{ij} + B_{ij} = B_{ij} + A_{ij} = (B + A)_{ij}.$$

   (c) If A and B are both $n \times n$ matrices, then $AB = BA$.

   Solution: False: Let

   $$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

   The

   $$AB = \begin{bmatrix} 10 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 1 \\ 0 & 0 \end{bmatrix} = A$$

   while

   $$BA = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 10 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}.$$

2. Let

   $$A = \begin{bmatrix} -2 & 1 & -2 \\ 1 & 0 & 2 \\ 3 & -3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \\ -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$

   Which of the following matrix multiplications are defined? Compute those which are defined.

   (a) AB

   Solution: We have

   $$AB = \begin{bmatrix} -2 & 1 & -2 \\ 1 & 0 & 2 \\ 3 & -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} -5 & 2 \\ 3 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 15 & -7 \\ -7 & 4 \\ -25 & 10 \end{bmatrix}.$$

(b) BC

Solution: We have

$$BC = \begin{bmatrix} -5 & 2 \\ 3 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 7 & -19 \\ -4 & 11 \\ 2 & -5 \end{bmatrix}.$$

(c) CA

Solution: We cannot multiply a 2×2 matrix by a 3×3 matrix.

(d) ABC

Solution: We can reuse our work from part (a)

$$(AB)C = \begin{bmatrix} 15 & -7 \\ -7 & 4 \\ -25 & 10 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -22 & 59 \\ 11 & -29 \\ 35 & -95 \end{bmatrix}.$$

3. Suppose that Math 54W is being taught by two different professors. Prof. A's lecture is more popular than Prof. B's lecture. In fact, each week 90% of A's students remain in the lecture, while only 10% switch into B's lecture. On the other hand, 20% of B's students switch into A's lecture, with 80% remaining in B's section.

   This situation is described in the following table:

	from A	from B
into A	90%	20%
into B	10%	80%

   which can be represented by the matrix

$$\begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$$

   Supposing that at the start of the semester each professor had 200 students, use matrix multiplication to answer the following:

   (a) How many students are there in each professor's section after the 1^{st} week? (Hint: represent the number of students in each section by a 2×1 column matrix.)

   Solution: We want to compute

$$\begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 200 \\ 200 \end{bmatrix} = \begin{bmatrix} 220 \\ 180 \end{bmatrix}.$$

   After the first week, 220 students are in Prof. A's class and 180 students are in Prof. B's class.

(b) How many students are there in each professor's section after the second week of classes?

Solution: Now we compute

$$\begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 220 \\ 180 \end{bmatrix} = \begin{bmatrix} 234 \\ 166 \end{bmatrix}.$$

After the second week, 234 students are in Prof. A's class and 166 students are in Prof. B's class.

4. True or false: If A and B are both invertible matrices, then $A + B$ is invertible.

Solution: False: Consider $A = I$ and $B = -I$, then $A + B$ is the zero matrix, and the zero matrix is not invertible.

5. Suppose A, B, C are invertible $n \times n$ matrices. What is

$$(ABC)^{-1}?$$

Solution: By our inverse rule for products,

$$((AB)C)^{-1} = C^{-1}(AB)^{-1} = C^{-1}B^{-1}A^{-1}.$$

We can verify directly that

$$(ABC)(C^{-1}B^{-1}A^{-1}) = (C^{-1}B^{-1}A^{-1})ABC = I.$$

6. (a) Find the inverse of

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Solution: We row reduce $[A \mid I]$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

and see

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

(b) Use part (a) to solve the system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 \qquad\quad + 8x_3 &= 17 \end{aligned}$$

Solution: Here

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

7. Let

$$T(x, y) = \begin{bmatrix} 4x - 6y \\ 2y \\ 3x \end{bmatrix}.$$

Is T invertible?

Solution: The function T is linear and can be represented by the matrix

$$A = \begin{bmatrix} 4 & -6 \\ 0 & 2 \\ 3 & 0 \end{bmatrix}.$$

Now this matrix is not square and so A^{-1} does not exist. Viewed another way, the function T does not map to $(1,1,1) \in \mathbf{R}^3$ and so T is not onto. We can see this in the following computation:

$$\left[\begin{array}{cc|c} 4 & -6 & 1 \\ 0 & 2 & 1 \\ 3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & -6 & 1 \\ 0 & 2 & 1 \\ 0 & 9/2 & 1/4 \end{array}\right]$$
$$\sim \left[\begin{array}{cc|c} 4 & -6 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{array}\right]$$
$$\sim \left[\begin{array}{cc|c} 4 & -6 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{cc|c} 4 & -6 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{cc|c} 4 & -6 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{cc|c} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

and we see our system is not consistent.

8. (a) What special properties does the matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ possess?

   Solution: For all $A \in \mathbf{R}^{2\times 2}$,

   $$AI = IA = A.$$

   (b) Given a 2×2 matrix A, can you always find another matrix B so that $AB = I$?

   Solution: No, consider

   $$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

then AB is the zero matrix.

(c) Given two 2×2 matrices A and B such that $AB = I$, is there anything noteworthy about BA?

Solution: We have a square matrix B such that $AB = I$ and thus A is invertible where $A^{-1} = B$ by (k) in the Invertible Matrix Theorem. Likewise $B^{-1} = A$ and so it follows that

$$BA = BB^{-1} = I = AB.$$

9. Compute $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n$. What is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

Solution: We have

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

while

$$A^3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

and in general

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

Thus

$$A^n \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+n \\ 1 \end{bmatrix}.$$

10. Square matrices A and B are said to *commute* if $AB = BA$. Find all 2×2 matrices which commute with:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

$$E = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad F = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}, \quad G = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}.$$

What patterns do you notice? For some of these it might help to notice that if A and B commute with M, then $A + B$ also commutes with M. What matrix always commutes with a square matrix M?