Discussion #2 1/26/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

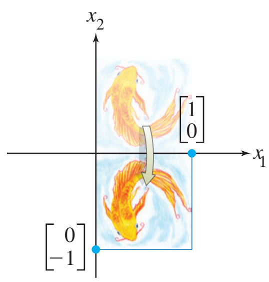
Problems

1. Suppose $T : \mathbf{R}^5 \to \mathbf{R}^2$ and $T(\mathbf{x}) = A\mathbf{x}$ for some matrix A and for each $\mathbf{x}$ in $\mathbf{R}^5$. How many rows and columns does A have?

   Solution: We require $A \in \mathbf{R}^{2\times 5}$, so 2 rows and 5 columns since $T : \mathbf{R}^5 \to \mathbf{R}^2$.

2. Let
$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$
   Give a geometric description of the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

   Solution: We have a reflection about the x_1-axis in $\mathbf{R}^2$.



3. Suppose
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 6x - 2y \\ 3 + y \end{bmatrix}$$
   Is T linear?

   Solution: No,
$$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$
   Linear transformations must map the zero vector to the zero vector.

4. Let
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$
   Define $T : \mathbf{R}^2 \to \mathbf{R}^2$ by mapping $\mathbf{x} \mapsto x_1\mathbf{v}_1 + x_2\mathbf{v}_2$.

   Find a matrix for T.

Solution: We have

$$T\mathbf{e}_1 = \begin{bmatrix} -2 \\ 5 \end{bmatrix} \quad \text{and} \quad T\mathbf{e}_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

so

$$A = \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix}$$

5. Find matrix that rotates $\mathbf{R}^3$ in the yz-plane by $\frac{\pi}{3}$ radians counterclockwise.

Solution: Consider

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\pi/3) & -\sin(\pi/3) \\ 0 & \sin(\pi/3) & \cos(\pi/3) \end{bmatrix}.$$

6. Suppose $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation

$$T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

Find $T\left(\begin{bmatrix} -1 \\ 11 \end{bmatrix}\right)$.

Solution: We want to express

$$\begin{bmatrix} -1 \\ 11 \end{bmatrix}$$

as a linear combination of

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Now

$$\left[\begin{array}{cc|c} 2 & 1 & -1 \\ 3 & -1 & 11 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1/2 & -1/2 \\ 0 & -5/2 & 25/2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -5 \end{array}\right]$$

and so

$$\begin{bmatrix} -1 \\ 11 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} - 5\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Then with T linear

$$T\left(\begin{bmatrix} -1 \\ 11 \end{bmatrix}\right) = 2T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) - 5T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -10 \\ -25 \end{bmatrix} = \begin{bmatrix} -8 \\ -23 \end{bmatrix}.$$

7. Why is the question "Is the linear transformation T onto?" an existence question?

Solution: For each $\mathbf{b}$, we are looking for a solution $\mathbf{x}$ to

$$T\mathbf{x} = \mathbf{b}.$$

8. Suppose $T : \mathbf{R}^4 \to \mathbf{R}^6$. Can T be onto?

   Solution: No, $A \in \mathbf{R}^{6\times 4}$ would have at most 4 pivots, which is less than the number of rows.

   Thus, the span of A's columns is not $\mathbf{R}^6$.