Discussion #4 1/30/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Introduction

Linear independence captures the idea that no vector in a collection can be written as a nontrivial combination of the others. A list of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset \mathbf{R}^n$ is linearly independent precisely when the only solution of

$$\sum_{j=1}^{k} c_j \mathbf{v}_j = \mathbf{0}$$

is the trivial solution $c_j = 0$ for all j. This concept identifies the smallest sets that span subspaces, underlies the notion of dimension, and guarantees uniqueness of solutions in homogeneous systems.

Problems

1. Let

$$A = \begin{bmatrix} 4 & 7 \\ 2 & k \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 29 \\ j \end{bmatrix}$$

satisfy

$$A \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \mathbf{b}.$$

Find j and k.

Solution: We have

$$A\mathbf{x} = \begin{bmatrix} 29 \\ 4 + 3k \end{bmatrix} = \begin{bmatrix} 29 \\ j \end{bmatrix}$$

and this implies

$$j = 4 + 3k$$

where $k \in \mathbf{R}$.

2. What is a surefire way to show two vectors are linearly dependent?

Solution: If

$$\mathbf{u} = k\mathbf{v}$$

for some $k \in \mathbf{R}$ the two vectors are linearly dependent.

3. Let

$$\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix}.$$

(a) Are the sets $\{\mathbf{u}, \mathbf{v}\}$, $\{\mathbf{u}, \mathbf{w}\}$, $\{\mathbf{u}, \mathbf{z}\}$, $\{\mathbf{v}, \mathbf{w}\}$, $\{\mathbf{v}, \mathbf{z}\}$, and $\{\mathbf{w}, \mathbf{z}\}$ each linearly independent? Why or why not?

Solution: Yes, no pair of vectors are multiples of each other.

(b) Does the answer to Part (a) imply that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ is linearly independent?

Solution: No, we have only shown subsets of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ with 2 or less objects are linearly independent.

(c) Is $\mathbf{z}$ a linear combination of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ (i.e, does $\mathbf{z}$ lie in Span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$)? Does this imply that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ are linearly independent?

Solution: $\mathbf{z}$ is not in the span. However, this does not imply that the vectors are linearly independent, as we shall see from the next part.

(d) Is $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ linearly dependent?

Solution: Yes, when we row reduce $[\mathbf{u} \quad \mathbf{v} \quad \mathbf{w} \quad \mathbf{z}]$ we will have at most 3 pivots but there are 4 columns. Thus there will be a nontrivial solution to

$$c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} + c_3\mathbf{z} = \mathbf{0}.$$

4. Find all $k \in \mathbf{R}$ such that

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ k \\ 1 \end{bmatrix}$$

are linearly independent.

Solution: We have

$$\left[\begin{array}{rrr|r} 1 & 3 & -1 & 0 \\ -1 & -5 & k & 0 \\ 4 & 7 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 3 & -1 & 0 \\ 0 & -2 & k-1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

so $R_2 \equiv R_3$, that is the two rows are equivalent, if and only if $k = 3$.

So the rows are linearly dependent if and only if $k = 3$. Therefore, the vectors are linearly independent exactly when $k \neq 3$.

We can also express this as, "If $k \neq 3$, we have a pivot in every column, and thus the vectors are linearly independent."

5. Construct a 4×3 matrix with linearly independent column vectors.

Does its column span $\mathbf{R}^4$?

Solution: Consider

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

then A does not have a pivot in every row and so A's column vectors do not span $\mathbf{R}^4$.

6.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Are A's columns linearly independent?

Solution: No, A's reduced row echelon form does not have a pivot in R_3. Hence A's columns are not linearly independent.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

7. Suppose $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{y} = 3\mathbf{b}$, where $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.

   What can you say about A's column vectors?

Solution: If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, then $\mathbf{x} \neq k\mathbf{y}$ for all $k \in \mathbf{R}$.

Thus,

$$\mathbf{x} - \frac{1}{3}\mathbf{y} \neq \mathbf{0} \in \mathbf{R}^n$$

and

$$A\left(\mathbf{x} - \frac{1}{3}\mathbf{y}\right) = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

Hence we have a nontrivial solution to $A\mathbf{x} = \mathbf{0}$.

This implies A's column vectors are linearly dependent.