

Discussion #33 4/24/26 – Spring 2026 MATH 54 Linear Algebra and Differential Equations

Problems

1. (a) Show

$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t},$$

where $c_1, c_2 \in \mathbf{R}$ satisfies

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}.$$

- (b) Find the eigenvalues of

$$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}.$$

- (c) Explain where the solution in (a) came from.

- (d) Find c_1 and c_2 such that

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

2. Consider the second order differential equation $y'' + 3y' + 4y = 0$.

- (a) Let $x_1 = y$ and $x_2 = y'$. Rewrite the equation as a system of two first order equations in the two functions x_1 and x_2 .

- (b) Show that the system in (a) can be written as $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix}.$$

- (c) Show that the equation $y''' + 2y'' - y' - 2y = 0$ can also be written as $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}.$$

3. Write

$$(1 - t^2)y'' - 2ty' + 2y = 0$$

in its normal form.

4. Consider the vectors

$$\mathbf{x}_1(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}, \quad \mathbf{x}_2(t) = \begin{bmatrix} e^t \\ 2t \end{bmatrix}.$$

- (a) Compute the Wronskian of \mathbf{x}_1 and \mathbf{x}_2 .

(b) Are \mathbf{x}_1 and \mathbf{x}_2 linearly independent on \mathbf{R} ?

5. Define $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, and $\mathbf{x}_3(t)$, for $-\infty < t < \infty$, by

$$\mathbf{x}_1(t) = \begin{bmatrix} \sin(t) \\ \sin(t) \\ 0 \end{bmatrix}, \quad \mathbf{x}_2(t) = \begin{bmatrix} \sin(t) \\ 0 \\ \sin(t) \end{bmatrix}, \quad \mathbf{x}_3(t) = \begin{bmatrix} 0 \\ \sin(t) \\ \sin(t) \end{bmatrix}$$

- (a) Show that for the three scalar functions *in each individual row* there are nontrivial linear combinations that sum to zero for all t .
- (b) Show that, nonetheless, the three vector functions are linearly independent. (No single nontrivial combination works for *each row, for all t* .)
- (c) Calculate the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3](t)$.