Discussion #24 3/30/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Problems

1. What can you say about the least squares solution of $A\mathbf{x} = \mathbf{b}$ when $\mathbf{b}$ is orthogonal to the columns of A?

2. Answer the following *True* or *False*. Justify each answer.

   (a) If $\mathbf{b}$ is in the column space of A, then every solution of $A\mathbf{x} = \mathbf{b}$ is a least-squares solution.

   (b) The least-squares solution of $A\mathbf{x} = \mathbf{b}$ is the point in the column space of A closest to $\mathbf{b}$.

   (c) A least-squares solution of $A\mathbf{x} = \mathbf{b}$ is a list of weights that, when applied to the columns of A, produces the orthogonal projection of $\mathbf{b}$ onto $\mathrm{Col}(A)$.

   (d) If $\hat{\mathbf{x}}$ is a least-squares solution of $A\mathbf{x} = \mathbf{b}$, then $\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}$.

3. Find the least squares solution $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}.$$

4. Let

$$A = \begin{bmatrix} 3 & 4 \\ -2 & 1 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 11 \\ -9 \\ 5 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

   Find $A\mathbf{u}$ and $A\mathbf{v}$ and compare them to $\mathbf{b}$. Will $\mathbf{u}$ be a possible least squares solution of $A\mathbf{x} = \mathbf{b}$?

5. Use the factorization $A = QR$ to find the least-squares solution of $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}.$$

6. Suppose $A \in \mathbf{R}^{m\times n}$ with n linearly independent column vectors and $\mathbf{b} \in \mathbf{R}^m$. Use the normal equations to find a formula for

$$\mathrm{proj}_{\mathrm{Col}(A)}(\mathbf{b}) = \mathbf{b}_{\mathrm{c}}$$

7. Find the equation of the line $y = mx + b$ that best fits the points $(-1, -1)$, $(1, 0)$, and $(2, 4)$ in the least-squares sense by following these steps:

(a) Write down the (inconsistent) system of three equations in two unknowns for this problem.

(b) Rewrite the system as a matrix equation $A\mathbf{x} = \mathbf{y}$, where $\mathbf{x} = \begin{bmatrix} m \\ b \end{bmatrix}$. What are A and $\mathbf{y}$?

(c) Find A^T and form the equation $A^T A\mathbf{x} = A^T\mathbf{y}$.

(d) Since the columns of A are linearly independent, $A^T A$ is invertible. Find $(A^T A)^{-1}$ and use it to solve the equation in part (c).

(e) What is the equation of the line that best fits the given points?

Midterm review

Let's recall some of the things we did before spring break.

1. Let $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$.

   (a) Find the eigenvalues of A and a basis for each eigenspace.

   (b) Diagonalize A.