Discussion #23 3/20/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Problems

1. What is the definition of a projection matrix?

2. Create an orthogonal projection matrix. Can you create a projection matrix that is not an orthogonal projection matrix?

3. Answer the following *True* or *False*. Justify each answer.

   (a) If W is a subspace of $\mathbf{R}^n$ and if $\mathbf{v}$ is in both W and $W^\perp$, then $\mathbf{v}$ must be the zero vector.

   (b) In the Orthogonal Decomposition Theorem, each term in

   $$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p}\mathbf{u}_p$$

   for $\hat{\mathbf{y}}$ is itself an orthogonal projection of $\mathbf{y}$ onto a subspace of W.

   (c) If $\mathbf{y} = \mathbf{z}_1 + \mathbf{z}_2$, where $\mathbf{z}_1$ is in a subspace W and $\mathbf{z}_2$ is in $W^\perp$, then $\mathbf{z}_1$ must be the orthogonal projection of $\mathbf{y}$ onto W.

   (d) The best approximation to $\mathbf{y}$ by elements of a subspace W is given by the vector $\mathbf{y} - \text{proj}_W \mathbf{y}$.

   (e) If an $n \times p$ matrix U has orthonormal columns, then $UU^T\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbf{R}^n$.

4. Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are orthogonal.

   Does it follow that $\mathbf{v}_1$, $\mathbf{v}_2$, and $c\mathbf{v}_3$ are orthogonal?

5. Suppose $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w} = 0$. Does it follow that $\mathbf{u} \cdot \mathbf{w} = 0$?

6. Find the closest point to $\mathbf{v}$ in the subspace W where

   $$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad W = \text{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

7. Let

   $$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

   Note that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal but that $\mathbf{u}_3$ is not orthogonal to $\mathbf{u}_1$ or $\mathbf{u}_2$. It can be shown that $\mathbf{u}_3$ is not in the subspace W spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$. Use this fact to construct a nonzero vector $\mathbf{v}$ in $\mathbf{R}^3$ that is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$.

8. Let

$$\mathbf{y} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$$

and $W = \text{Span}\,\{\mathbf{u}_1\}$.

(a) Let U be the 2×1 matrix whose only column is $\mathbf{u}_1$. Compute $U^T U$ and UU^T.

(b) Compute $\text{proj}_W(\mathbf{y})$ and $(UU^T)\mathbf{y}$.

9. Let W be a subspace of $\mathbf{R}^n$. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbf{R}^n$ and let $\mathbf{z} = \mathbf{x} + \mathbf{y}$. If $\mathbf{u}$ is the projection of $\mathbf{x}$ onto W and $\mathbf{v}$ is the projection of $\mathbf{y}$ onto W, show that $\mathbf{u} + \mathbf{v}$ is the projection of $\mathbf{z}$ onto W.

10. Answer the following *True* or *False*. Justify each answer.

(a) If $W = \text{Span}\,\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ with $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ linearly independent, and if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set in W, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for W.

(b) If $\mathbf{x}$ is not in a subspace W, then $\mathbf{x} - \text{proj}_W(\mathbf{x})$ is not zero.

(c) In a QR factorization, say $A = QR$ (when A has linearly independent columns), the columns of Q form an orthonormal basis for the column space of A.

11. Use the Gram-Schmidt process to transform the basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ into an orthonormal basis.

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

12. Find the QR factorization for

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

given

$$Q = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}.$$

13. Find the QR factorization for

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}.$$

Hint: The column vectors appeared in the lecture on §6.4 - The Gram-Schmidt Process.

14. Suppose $A = QR$, where $Q \in \mathbf{R}^{m \times n}$ and $R \in \mathbf{R}^{n \times n}$.

If A has linearly independent column vectors, prove that R must be invertible.

Note that A need not be square.