Discussion #21 3/16/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Problems

1. Answer the following *True* or *False*. Explain your reasoning, or give a counterexample.
   (a) If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbf{R}^4$ such that $\mathbf{x} \cdot \mathbf{y} = 0$, then either $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$.
   (b) If $\mathbf{u} \neq \mathbf{0}$ is a nonzero vector in $\mathbf{R}^n$ then the vector $\frac{\mathbf{u}}{\|\mathbf{u}\|}$ has norm 1.
   (c) If A is a 3×3 matrix, and $\mathbf{x}$ and $\mathbf{y}$ are column vectors in $\mathbf{R}^3$, then
   $$A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A\mathbf{y}.$$
   (d) If A is an $m \times n$ matrix, the equation $A\mathbf{x} = \mathbf{y}$ has a solution if and only if $\mathbf{y}$ is in the column space of A.
   (e) If A is an $m \times n$ matrix whose columns are linearly independent, then $A^T A$ is invertible.
   (f) Any basis for $\mathbf{P}_n$ (the vector space of all polynomials of degree $\leq n$) must contain a polynomial of degree k for each $k = 0, 1, 2, \ldots, n$.
   (g) Let $\mathcal{B} = \{\mathbf{u}_1, \ldots \mathbf{u}_n\}$ be an orthogonal set in $\mathbf{R}^m$. Then
   $$U = \begin{bmatrix} \mathbf{u}_1 & \ldots & \mathbf{u}_p \end{bmatrix}$$
   satisfies
   $$U^T U = I_n.$$
   (h) If A is a 4×4 matrix whose column vectors form an orthonormal basis for $\mathbf{R}^4$, then A is invertible.

2. If $\mathbf{a}, \mathbf{b} \in \mathbf{R}^n$, what is the difference between
$$\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \quad \text{and} \quad \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a}?$$

3. Let
$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}.$$
   (a) Compute $\|\mathbf{u} + \mathbf{v}\|$ and $\|\mathbf{u}\| + \|\mathbf{v}\|$. Which is larger?
   (b) Compute $\|\mathbf{v} \cdot \mathbf{w}\|$ and $\|\mathbf{v}\|\,\|\mathbf{w}\|$. Which is larger?
   (c) Compute $\mathbf{u}^T\mathbf{v}$, $\mathbf{v}^T\mathbf{u}$, and $\mathbf{u} \cdot \mathbf{v}$. Is $\mathbf{v}\mathbf{u}^T$ defined? If so, compute it. If not, explain.

4. Let W be a subspace of $\mathbf{R}^n$. Define the **orthogonal complement** of W to be the subspace
$$W^{\perp} = \{\mathbf{v} \in \mathbf{R}^n : \mathbf{w} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{w} \in W\}.$$

(a) Consider the subspace $W = \text{Span}\{(1,1,1),(2,0,-1)\}$ of $\mathbf{R}^3$. Find a vector which spans $W^\perp$.

(b) Express the vector $(2,1,-3)$ in the form $\mathbf{w}+\mathbf{w}^\perp$, where $\mathbf{w}\in W$ and $\mathbf{w}^\perp \in W^\perp$.

5. Let

$$W = \text{Span}\{(1,2,3)\}.$$

Find a basis for $W^\perp$.

6. Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has the property that

$$\|A\mathbf{x}\| = \|\mathbf{x}\|$$

for all $\mathbf{x} \in \mathbf{R}^2$.

(a) Construct 3 different matrices that have the same properties as A. What do your 3 matrices all have in common?

(b) Compute $\|A^2\mathbf{x}\|$.

(c) What are the possible eigenvalues for A?

(d) Can A be symmetric?

(e) Must A be symmetric?

(f) If A^{-1} exists, compute $\|A^{-1}\mathbf{x}\|$.

(g) Must A be invertible?

7. (a) Let θ be a real number. Show that the vectors $\mathbf{v}_1 = (\cos(\theta),\sin(\theta))$ and $\mathbf{v}_2 = (-\sin(\theta),\cos(\theta))$ form an orthonormal basis for $\mathbf{R}^2$.

(b) Let $\mathbf{u}_1 = (1,1)$ and $\mathbf{u}_2 = (0,-1)$. Find a 2×2 matrix A which induces a linear transformation T_A such that $T_A(\mathbf{u}_1) = \mathbf{v}_1$ and $T_A(\mathbf{u}_2) = \mathbf{v}_2$.

8. (a) Show that $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is an orthonormal basis for $\mathbf{R}^3$, where $\mathbf{v}_1 = \left(1/\sqrt{2}, 1/\sqrt{2}, 0\right)$, $\mathbf{v}_2 = \left(-1/\sqrt{2}, 1/\sqrt{2}, 0\right)$, and $\mathbf{v}_3 = (0,0,1)$.

(b) Find the coordinates of $\mathbf{w} = \left(\sqrt{2}, 3\sqrt{2}, -4\right)$ with respect to this basis.

(c) Let

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Show that $A\mathbf{v}_1 = 2\mathbf{v}_1$, $A\mathbf{v}_2 = 4\mathbf{v}_2$, and $A\mathbf{v}_3 = -\mathbf{v}_3$.

(d) Compute $A\mathbf{w}$ directly, and again by using your work from parts (b) and (c).

9. A matrix is **orthogonal** if it is square and its columns are orthonormal.

Let A be a 2×2 orthogonal matrix.

(a) Show that A is of the form

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix},$$

for some value of θ in the interval $0 \leq \theta < 2\pi$.

(b) Geometrically, what does an orthonormal basis for $\mathbf{R}^2$ look like?

(c) What does an orthonormal basis for $\mathbf{R}^3$ look like?