Discussion #20 3/13/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Problems

1. Answer the following *True* or *False*. Explain your reasoning, or give a counterexample.

   (a) The vectors
   $$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1+2i \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 6i \\ -6-3i \end{bmatrix}$$
   are linearly independent in $\mathbf{C}^2$.

   (b) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation that rotates the plane by θ radians counterclockwise. Then T's matrix representation is **not** diagonalizable.

   (c) If $B = P^{-1}AP$ and $\mathbf{x}$ is an eigenvector of A corresponding to an eigenvalue λ, then $P^{-1}\mathbf{x}$ is an eigenvector of B corresponding also to λ.

   (d) Let A be a complex (or real) $n \times n$ matrix, and let $\mathbf{x} \in \mathbf{C}^n$ be an eigenvector corresponding to an eigenvalue $\lambda \in \mathbf{C}$.

   Then for each nonzero complex scalar μ, the vector $\mu\mathbf{x}$ is an eigenvector of A.

2. Let $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be bases for vector spaces V and W, respectively.

   Let $T : V \to W$ be a linear transformation with the property that
   $$T(\mathbf{d}_1) = 2\mathbf{b}_1 - 3\mathbf{b}_2, \quad T(\mathbf{d}_2) = -4\mathbf{b}_1 + 5\mathbf{b}_2.$$
   Find the matrix for T relative to $\mathcal{D}$ and $\mathcal{B}$.

3. Find $T(a_0 + a_1 t + a_2 t^2)$, if T is the linear transformation from $\mathbf{P}_2$ to $\mathbf{P}_2$ whose matrix relative to $\mathcal{B} = \{1, t, t^2\}$ is
   $$[T]_{\mathcal{B}} = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}.$$

4. Find the complex-valued eigenvalues of
   $$A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$$
   and a basis for each eigenspace in $\mathbf{C}^2$.

5. Determine if
   $$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$
   are similar.

6. Prove the following: If A is invertible and similar to B, then B is invertible and A^{-1} is similar to B^{-1}.

7. Define $T : \mathbf{R}^2 \to \mathbf{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}.$$

Find a basis $\mathcal{B}$ for $\mathbf{R}^2$ with the property that $[T]_{\mathcal{B}}$ is diagonal.