

Discussion #18 3/9/26 – Spring 2026 MATH 54

Linear Algebra and Differential Equations

Problems

- For each of the following matrices, describe in geometric terms the eigenspaces (if any) and their associated eigenvalues. *Do not compute the matrices.*
 - The matrix induced by the linear transformation $T : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ which reflects each vector across the z -axis.
 - The matrix induced by the linear transformation $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ which rotates each vector by $\pi/4$ radians counterclockwise.
- Let A be an $n \times n$ matrix. Show that A is invertible if and only if 0 is not an eigenvalue of A .
- Answer the following *True* or *False*. Explain your reasoning, or give a counterexample.
 - Any $n \times n$ matrix that has fewer than n real distinct eigenvalues is not diagonalizable.
 - Eigenvectors corresponding to the same eigenvalue are always linearly dependent.
 - If A is diagonalizable, then it has at least one eigenvalue.

- Show that if $b \neq 0$ then

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

is not diagonalizable.

- Find the eigenvalues, and bases for the associated eigenspaces, of

$$A = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 0 & 0 & 2 \end{bmatrix}.$$

- Diagonalize the matrix from part (a).
- Give an example of a 2×2 matrix with two linearly independent eigenvectors, but only one eigenvalue.
- Find a 3×3 matrix with eigenvalues $0, 1, -1$ and corresponding eigenvectors $(0, 1, 1)$, $(1, -1, 1)$, and $(0, 1, -1)$.

- Is

$$\begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$$

diagonalizable?

9. (a) Suppose that $P^{-1}AP = D$, where D is diagonal. Show that $A^k = PD^kP^{-1}$ for any positive integer k .
- (b) Use part (a) to compute A^{10} , where

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}.$$

10. Two $n \times n$ matrices A and B are said to be *similar* if there is an invertible $n \times n$ matrix P such that $B = P^{-1}AP$. Show that similar matrices always have the same eigenvalues. Must they have the same eigenvectors?