Discussion #17 3/6/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Eigenvectors and eigenvalues play an important role in computational mathematics. Eigenvectors have a simple description, we need a nonzero vector $\mathbf{v} \in \mathbf{R}^n$ such that for $A \in \mathbf{R}^{n\times n}$ we have

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some $\lambda \in \mathbf{R}$. Eigenvectors are not just limited to vector spaces such as $\mathbf{R}^n$, the differentiation operator has its own eigenfunctions and they play a fundamental role in a later coursework.

But before we get into that, let's do a little change of basis stuff.

Questions

1. Let $\mathbb{R}^2$ have the standard basis $\mathcal{E} = \{\mathbf{e_1}, \mathbf{e_2}\}$ and a non-standard basis $\mathcal{B} = \{\mathbf{b_1}, \mathbf{b_2}\}$, where

$$\mathbf{b_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{b_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

   (a) Find the change of basis matrix $P_{\mathcal{E}\leftarrow\mathcal{B}}$.

   (b) Find the change of basis matrix $P_{\mathcal{B}\leftarrow\mathcal{E}}$.

   (c) If a vector $\mathbf{x}$ has coordinates $[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, find $[\mathbf{x}]_E$.

2. Consider two bases for $\mathbb{R}^2$, $\mathcal{B} = \{\mathbf{u_1}, \mathbf{u_2}\}$ and $\mathcal{C} = \{\mathbf{v_1}, \mathbf{v_2}\}$, defined by:

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   (a) Find the change of basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$.

   (b) Find the change of basis matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$.

   (c) If $\mathbf{w} = \mathbf{u_1} + 2\mathbf{u_2}$, what is $[\mathbf{w}]_{\mathcal{C}}$?

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation whose matrix relative to the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$ is:

$$[T]_{\mathcal{E}} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

   Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^2$.

   (a) Find the change of basis matrix P from $\mathcal{B}$ to $\mathcal{E}$.

   (b) Use the similarity transformation formula $[T]_{\mathcal{B}} = P^{-1}[T]_{\mathcal{E}}P$ to find the matrix of T relative to the basis $\mathcal{B}$.

4. Answer the following *True* or *False*. Explain your reasoning, or give a counterexample.
   (a) The sum of two eigenvalues of a matrix A is also an eigenvalue of A.
   (b) The sum of two eigenvectors of a matrix A is also an eigenvector of A.
   (c) There exists a square matrix with no real eigenvalues.
   (d) There exists an $n \times n$ matrix with $n+1$ distinct eigenvalues.

5. Suppose D is the differentiation operator on the space of differentiable functions over $\mathbf{R}$,

$$Df = \frac{d}{dx} f(x) = f'(x).$$

   Can you find a function f such that

$$f \neq 0 \quad \text{and} \quad Df = \lambda f$$

   for some $\lambda \in \mathbf{R}$? Is your function f unique?

6. Given $\lambda = 6$ and $\lambda = -1$ are eigenvalues of

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}$$

   find a basis for each eigenspace of A.

7. Is 5 an eigenvalue of

$$A = \begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix}?$$

8. Find the characteristic polynomial and the eigenvalues of the matrix

$$\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}.$$

9. List the eigenvalues, repeated according to their multiplicities, of the matrix

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 1 & -5 & 2 & 1 \end{bmatrix}.$$

10. Show that if A^2 is the zero matrix, then the only eigenvalue of A is 0. Must A be the zero matrix?