Discussion #8 2/9/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

The Invertible Matrix Theorem, Lay 2.3 Theorem 8 (p. 145), is very very useful. You don't need to memorize all the parts of it, but you should know what they mean.

Determinants can be confusing, but once you learn to calculate them and a few of their basic properties, you'll know all you need to know.

Problems

1. Let A be a square matrix. List at least four different statements which are equivalent to the statement
$$\det(A) \neq 0.$$

2. Let A and B be $n \times n$ matrices. Answer the following *True* or *False*. If *False* give a counterexample.
   (a) $\det(AB) = \det(A) \cdot \det(B)$
   (b) $\det(AB) = \det(BA)$
   (c) $\det(A + B) = \det(A) + \det(B)$
   (d) If A and B are both invertible, then AB is invertible.
   (e) $\det(A) = \det\left(A^T\right)$
   (f) If A is invertible, then $\det(A^{-1}) = (\det(A))^{-1}$.
   (g) If c is a scalar then $\det(cA) = c^n \det(A)$.

3. What is the Area of a parallelogram with vertices at $(0, 0)$, $(4, 1)$, $(3, 5)$, $(7, 6)$?

4. Suppose A is a 3×3 matrix with determinant 5. What is $\det(3A)$? $\det(A^{-1})$? $\det(2A^{-1})$? $\det((2A)^{-1})$?

5. Compute the following:

$$\text{(a)}\quad \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} \qquad \text{(b)}\quad \begin{vmatrix} 1 & -1 & -2 \\ 3 & 0 & 1 \\ -1 & 1 & 1 \end{vmatrix} \qquad \text{(c)}\quad \begin{vmatrix} \sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

6. By inspection, evaluate the following determinants:

$$\text{(a)}\quad \begin{vmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 5 & 9 \end{vmatrix} \qquad \text{(b)}\quad \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 9 \\ 0 & 16 & 25 & 36 \\ 49 & 64 & 81 & 100 \end{vmatrix} \qquad \text{(c)}\quad \begin{vmatrix} 0 & \pi & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 22 \end{vmatrix}$$

7. Show that if A is a square matrix with a row of zeros, then $\det(A) = 0$. What if A has a column of zeros?

8. (a) Show that the equation $A\mathbf{x} = \mathbf{x}$ can be rewritten as

$$(A - I)\,\mathbf{x} = 0,$$

where A is an $n \times n$ matrix and I is the $n \times n$ identity matrix.

(b) Use part (a) to solve $A\mathbf{x} = \mathbf{x}$ for $\mathbf{x}$, where

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & -1 \\ 2 & -2 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(c) Solve $A\mathbf{x} = 4\mathbf{x}$.