Discussion #7 2/6/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Problems

1. Answer the following with *True* or *False*. Explain your reasoning, or give a counterexample.
   (a) If A and B are any matrices, then $A + B$ is defined.
   (b) If A and B are both $n \times n$ matrices, then $A + B = B + A$.
   (c) If A and B are both $n \times n$ matrices, then $AB = BA$.

2. Let
$$A = \begin{bmatrix} -2 & 1 & -2 \\ 1 & 0 & 2 \\ 3 & -3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \\ -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$
   Which of the following matrix multiplications are defined? Compute those which are defined.
   (a) AB
   (b) BC
   (c) CA
   (d) ABC

3. Suppose that Math 54W is being taught by two different professors. Prof. A's lecture is more popular than Prof. B's lecture. In fact, each week 90% of A's students remain in the lecture, while only 10% switch into B's lecture. On the other hand, 20% of B's students switch into A's lecture, with 80% remaining in B's section.

   This situation is described in the following table:

	from A	from B
into A	90%	20%
into B	10%	80%

   which can be represented by the matrix
$$\begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$$
   Supposing that at the start of the semester each professor had 200 students, use matrix multiplication to answer the following:
   (a) How many students are there in each professor's section after the 1^{st} week? (Hint: represent the number of students in each section by a 2×1 column matrix.)
   (b) How many students are there in each professor's section after the second week of classes?

4. True or false: If A and B are both invertible matrices, then $A+B$ is invertible.
5. Suppose A, B, C are invertible $n \times n$ matrices. What is
$$(ABC)^{-1}?$$
6. (a) Find the inverse of
$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$
   (b) Use part (a) to solve the system
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 \qquad\quad + 8x_3 &= 17 \end{aligned}$$
7. Let
$$T(x, y) = \begin{bmatrix} 4x - 6y \\ 2y \\ 3x \end{bmatrix}.$$
   Is T invertible?
8. (a) What special properties does the matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ possess?
   (b) Given a 2×2 matrix A, can you always find another matrix B so that $AB = I$?
   (c) Given two 2×2 matrices A and B such that $AB = I$, is there anything noteworthy about BA?
9. Compute $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n$. What is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$?
10. Square matrices A and B are said to *commute* if $AB = BA$. Find all 2×2 matrices which commute with:
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$
$$E = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad F = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}, \quad G = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}.$$
   What patterns do you notice? For some of these it might help to notice that if A and B commute with M, then $A+B$ also commutes with M. What matrix always commutes with a square matrix M?