Discussion #4 1/30/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

Introduction

Linear independence captures the idea that no vector in a collection can be written as a nontrivial combination of the others. A list of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset \mathbf{R}^n$ is linearly independent precisely when the only solution of

$$\sum_{j=1}^{k} c_j \mathbf{v}_j = \mathbf{0}$$

is the trivial solution $c_j = 0$ for all j. This concept identifies the smallest sets that span subspaces, underlies the notion of dimension, and guarantees uniqueness of solutions in homogeneous systems.

Problems

1. Let

   $$A = \begin{bmatrix} 4 & 7 \\ 2 & k \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 29 \\ j \end{bmatrix}$$

   satisfy

   $$A \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \mathbf{b}.$$

   Find j and k.

2. What is a surefire way to show two vectors are linearly dependent?

3. Let

   $$\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix}.$$

   (a) Are the sets $\{\mathbf{u}, \mathbf{v}\}$, $\{\mathbf{u}, \mathbf{w}\}$, $\{\mathbf{u}, \mathbf{z}\}$, $\{\mathbf{v}, \mathbf{w}\}$, $\{\mathbf{v}, \mathbf{z}\}$, and $\{\mathbf{w}, \mathbf{z}\}$ each linearly independent? Why or why not?

   (b) Does the answer to Part (a) imply that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ is linearly independent?

   (c) Is $\mathbf{z}$ a linear combination of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ (i.e, does $\mathbf{z}$ lie in $\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$)? Does this imply that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ are linearly independent?

   (d) Is $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ linearly dependent?

4. Find all $k \in \mathbf{R}$ such that

   $$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ k \\ 1 \end{bmatrix}$$

   are linearly independent.

5. Construct a 4×3 matrix with linearly independent column vectors.

   Does its column span $\mathbf{R}^4$?

6.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

   Are A's columns linearly independent?

7. Suppose $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{y} = 3\mathbf{b}$, where $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.

   What can you say about A's column vectors?